Philosophy 500 — June 9: Quantifier logic

Recall from before:

(1) Some A's aren't B's	(1) $\exists x(Ax \,\&\, \neg Bx)$
(2) Not all A's are B's	(2) $\neg\forall x(Ax \rightarrow Bx)$
(3) All A's are B's	(3) $\forall x(Ax \rightarrow Bx)$
(4) Some A's are B's	(4) $\exists x(Ax \,\&\, Bx)$
(5) No A's are B's	(5) $\neg\exists x(Ax \,\&\, Bx)$
(6) Every A isn't a B	(6) $\forall x(Ax \rightarrow \neg Bx)$

(1) and (2) are logically equivalent to each other, and are the negation of (3).
(5) and (6) are logically equivalent to each other, and are the negation of (4).

We will use the same key as before:

UD: all people	Bx: x is a bouncer.	Ixyz: x introduced y to z.
a: Alice	Mx: x is a merchant	Kxy: x knows y.
b: Betty	Px: x is a plumber.	Lxy: x likes y.
c: Chris	Tx: x is tall.	Oxy: x is older than y.

Syntax (Formal rules of the quantifier logic language)

We will want to be able to say what is and is not allowed in using our symbolic language. Before this, though, we need some new concepts. What is said with x should be taken to apply to the other variables as well (e.g. y-quantifier is defined accordingly, etc.).

The **scope of a quantifier** is what is inside the parentheses that follow that quantifier.
An **x-quantifier** is either $\forall x$ or $\exists x$.
An instance of x is called a **bound instance** if it is inside the scope of an x-quantifier.
An instance of x is called a **free instance** if it isn't inside the scope of an x-quantifier.
x is **free in a formula** if the formula contains no x-quantifiers.
x is **bound in a formula** if formula contains x-quantifiers.

The scope is very important. To see this, notice that the following sentences don't mean the same thing:

$\forall x(Px \rightarrow Bc)$
For each person, if that person is a plumber, then Chris is a bouncer.
If someone is a plumber, then Chris is a bouncer.
$\forall x(Px) \rightarrow Bc$
If each person is a plumber, then Chris is a bouncer.
If everyone is a plumber, then Chris is a bouncer.

As in sentential logic, we will define what is and is not proper by starting with the basic components and building up using rules.

A **proper formula** is one that's constructed using the rules of the language, and is completely unambiguous as to its meaning.

These rules are:

1. A predicate together with the right number of inputs (constants or variables) is a proper formula.
2. If $\mathcal{A}$ is a formula, so is $\neg\mathcal{A}$.
3. If $\mathcal{A}$ and $\mathcal{B}$ are proper formulas, so are $(\mathcal{A} \to \mathcal{B}), (\mathcal{A} \leftrightarrow \mathcal{B}), (\mathcal{A} \,\&\, \mathcal{B})$, and $(\mathcal{A} \vee \mathcal{B})$.
4. If $\mathcal{A}$ is a proper formula which doesn't contain any x-quantifiers, then $\forall x(\mathcal{A})$ and $\exists y(\mathcal{A})$ are also proper formulas.

Anything else is not a proper formula. In particular, a predicate with the wrong number of inputs, an x-quantifier in the scope of another x-quantifier, a quantifier with a constant instead of a variable, etc. is not allowed. This means that whether or not something is a proper formula can only be decided if we have the key so we can know what the predicates and constants are and how many inputs each predicate needs.

A **proper sentence** is a proper formula in which there are no free instances of any variable.

A proper sentence is the sort of thing which could be true or false (which matches our previous definition).

A proper formula, on the other hand, could be something like Px, which can't be true or false because x is a variable, so it lacks meaning by itself.

But we need to define 'proper formula' in order to define 'proper sentence'.

Every expression in quantified logic is one of these:
(a) Not a proper formula (and, so, not a proper sentence either).
(b) A proper formula, but not a proper sentence.
(c) A proper sentence (and, so, also a proper formula).

The identity predicate

There are still some sentences we can't translate. For example: "Betty is the only plumber".

We could rephrase is as "Betty is a plumber, and no one other than Betty is a plumber". But we still can't translate the second half, because we have no way of saying something like "x isn't Betty". So we'll introduce a new operator, called **the identity operator**

Identity predicate: $x = y$ **means** x *is* y.

Examples

1. Betty is the only plumber
   A person is a plumber if and only if it's Betty.
   For every person, that person is a plumber if and only if it's Betty.
   $\forall x(Px \leftrightarrow x = b)$

2. No tall person other than Chris is older than Betty.
   No one who is tall and isn't Chris is older than Betty.
   $\neg\exists x((Tx \,\&\, \neg x = c) \,\&\, Oxb)$
3. Some plumber knows everyone except Chris.
   There is a plumber who knows everyone who isn't Chris and doesn't know Chris.
   There is a plumber for whom the only person they don't know is Chris.
   $\exists x(Px \,\&\,$ (the only person x doesn't know is Chris))
   $\exists x(Px \,\&\, \forall y(Kxy \leftrightarrow \neg y = c))$
4. There's at most one plumber.
   If a person is a plumber, then no other person is.
   $\forall x(Px \rightarrow \forall y(\neg x = y \rightarrow \neg Py))$
   $\forall x(Px \rightarrow \forall y(Py \rightarrow x = y))$
5. There's exactly one plumber.
   Someone is a plumber, and there's at most one plumber.
   $\exists x(Px) \,\&\, \forall x(Px \rightarrow \forall y(Py \rightarrow y = x))$
   There's a person such that they're the only plumber.
   $\exists x(\forall y(Py \leftrightarrow y = x))$

Notice this crucial difference:

"Betty is the only person who isn't a plumber" affirms Betty isn't a plumber
"Everyone other than Betty is a plumber" leaves open whether Betty is a plumber.

Exercises

UD: all people	Bx: x is a bouncer.	Ixyz: x introduced y to z.
a: Alice	Mx: x is a merchant	Kxy: x knows y.
b: Betty	Px: x is a plumber.	Lxy: x likes y.
c: Chris	Tx: x is tall.	Oxy: x is older than y.

1. Every merchant other than Alice knows a tall plumber.

2. If anyone is a tall merchant, it's Chris.

3. Chris isn't the only bouncer, but he is the only tall bouncer.

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4. Betty likes every merchant who's older than every plumber.

5. Not every merchant who has been introduced to Chris likes him.

6. Betty knows the only plumber.

7. No merchant whom Betty knows is older than every plumber, unless it's Chris.

8. Betty is the only person who hasn't been introduced to Alice.

9. Alice and Betty are the only two plumbers.

Homework #8, Due June 14, 2010

A. Translate each of the following sentences in the space provided, using the key given. Most of these are complicated, so I recommend doing the translation in steps on scratch paper first. (2 pts. each)

UD: all dogs	Px: x is a pomeranian.	Lxyz: x likes y better than z.
f: Fido	Cx: x is a cockapoo.	Bxy: x has bitten y.
e: Eric	Fx: x can fetch.	Oxy: x is older than y.
k: Kara	Hx: x is hairy.	Sxy: x smells worse than y.

1. Every dog other than Fido and Eric can fetch.

2. If any pomeranian older than Fido is hairy, it's Kara.

3. Every cockapoo smells worse than Kara, unless it's bitten a pomeranian.

4. Eric is the only hairy cockapoo, but Fido is also a hairy cockapoo.

5. Some cockapoo likes Fido better than it likes any dog which smells worse than Kara.

6. Only hairy pomeranians smell worse than every cockapoo.

7. No cockapoo which has bitten Kara is older than even one hairy dog.

8. Some pomeranian hasn't bitten any dogs, but smells worse than all dogs other than itself.

9. Fido has bitten the only hairy cockapoo, and it has also bitten Fido.

UD: all dogs	Px: x is a pomeranian.	Lxyz: x likes y better than z.
f: Fido	Cx: x is a cockapoo.	Bxy: x has bitten y.
e: Eric	Fx: x can fetch.	Oxy: x is older than y.
k: Kara	Hx: x is hairy.	Sxy: x smells worse than y.

B. For each of the following, indicate whether it's not a proper formula (NF), a proper formula but not a proper sentence (F), or a proper sentence (S). (1 pt. each)

1. $\forall x(Lfx \rightarrow Sx)$

2. $Hx \rightarrow \forall x(Hx \vee Fx)$

3. $\neg\exists f(Fx \rightarrow Bfx)$

4. $\forall x(\neg\exists y(xf \mathbin{\&} \neg Oxy))$

5. $\forall z(Oez \rightarrow Bz \vee Lz)$

6. $\forall y[(Hy \mathbin{\&} Sey) \rightarrow \neg\exists x(Lxky)]$

7. $\neg\forall y[Cy \vee \exists x(\neg Shx \vee Fy)]$

B. For each of the following, state whether it's true or false. If it's true, explain **in paragraph form and in detail** how you can be sure of that. If it's false, give a counterexample (i.e. an example which shows it's false). (2 pts. each)

1. If $A \leftrightarrow (B \vee C)$ is a tautology, so is $B \leftrightarrow C$.

2. If $B \mathbin{\&} C$ is contingent, either B or C is contingent.

3. If a valid argument has contingent premises, it's conclusion is contingent.

4. If $A, B \rightarrow C \therefore D$ is valid, then $\{A, B, C, D\}$ is inconsistent.

5. If a formula contains a free instance of x in it, then it's allowed to enclose it in parentheses and add an x-quantifier in front.