Philosophy 500 — June 2: Quantifier logic

With sentential logic, we have been taking sentences as our basic logical units. We would translate the sentence "Alice went to the beach and Bob went to the beach" as $A \& B$, with the A standing for "Alice went to the beach" and the B for "Bob went to the beach", even though it's gramatically just one sentence. But it's a compound sentence, so that we're still not going further than taking sentences as the basic building blocks. The same goes for "Alice and Bob went to the beach": though it's not a compound sentence, we paraphrase it as "Alice went to the beach and Bob went to the beach", which is.

With quantifier logic, we're going to take two additional steps:

1. Analyze sentences into predicates and subjects/objects.
2. Introduce the quantifiers "there exists" and "for all".

Step 1: Predicates, constants, and variables

We will denote predicates with upper case letters, and the subjects/objects of these with lower case letters. We'll drop the distinction between subject and object, and just refer to them all as the predicate's **inputs**. The word 'arguments' is often used to mean the same as 'inputs' in this context, but we'll avoid that, to avoid confusion with arguments in the sense we've already talked about.

You can think of a predicate as a sentence with certain parts removed, and these parts are then filled in by the inputs. We'll be dealing with two types of inputs: constants and variables. **For constants we'll use the letters a, b, c and so on, but we'll stop at v. For variables we'll use the letters x, y, z, w.** Roughly speaking, constants have meaning on their own, whereas variables have no fixed meaning. Variables are only really useful with quantifiers (i.e. at step 2), but we can read sentences with variables even without quantifiers... they just won't have concrete meaning. As usual, we need to give a translation key if we want our sentences to have concrete meaning. In defining the predicates, we'll use variables as stand-ins for any kind of inputs (e.g. "Ax: x is an ant"). Here's an example:

Ax: x is an ant.
Bxy: x bartered with y.
Txy: x is taller than y.
Cx: Bill Clinton knows x.
a: Alex
b: Binky the Clown
c: Charles Barkley

Notice that in the definition of A the input is the sentence's subject, whereas in that of C the input is the object, and in that of B one input is the object and the other the subject. We can see, then, why it's useful to just call them all inputs instead of worrying about these things. Also, in predicates with more than one input, the order that they're listed in is important. For example, Byx means "y bartered with x", Bca means "Charles Barkley bartered with Alex", and so on. We can now translate some sentences:

"Alex and Binky the Clown are ants, but Alex didn't barter with himself" would be $(Aa \& Ab) \& \neg Baa$.

However, we need to restrict what we can use as a constant. For example, if we added to the key "d: dogs", then the sentence Ad would read as "Dogs is an ant", which isn't very good. So we're going to require that our constants refer to a single thing. This can be done either by name (e.g. Alex) or by description (e.g. The current Queen of England). But we must make sure that if we're referring to a thing by description it's actually referring to something and to only that thing. For example "Or Neeman's student" is not a proper term, because I have more than one, and "The present king of France" is not proper because there is no king of France. We'll come to this problem later, but for now we just need to make sure we don't use non-proper terms.

Exercises

Using the key above, translate each of the following sentences:

1. Charles Barkley is an ant, but Alex isn't.

2. Bill Clinton knows Alex only if Charles Barkley bartered with y.

3. Bill Clinton knows Alex and Binky the Clown, both of whom are ants, but neither of which bartered with Charles Barkley.

4. Alex is shorter than Charles Barkley.

5. Alex and Binky the Clown are the same height.

6. If x is an ant, then Bill Clinton knows it.

Step 2: Quantifiers

The problem is that we still can't translate a lot of sentences. For example: "Dogs are mammals", "Everybody has to die", etc. For these we'll need to introduce quantifiers. We'll use the symbol $\forall$ to mean "for all", and the symbol $\exists$ to mean "there exists". However, just like these phrases, the symbols by themselves are not very meaningful. We're going to write a variable following each quantifier, followed by an expression in parentheses (which will, typically, involve that variable) which is called the quantifier's **scope**:

$\forall$x(Ax) reads as "For all x, Ax". Using the previous key, this would mean "For all x, x is an ant", or, in other words, "Everything is an ant". Likewise:

$\exists$x(Ax) reads as "There exists an x such that Ax". Using the key, this would mean "There exists an x such that x is an ant", or "There exists an ant".

We can also have more complicated sentences instead of Ax. For example $\forall x(Ax \rightarrow Cx)$ reads as "For all x, if x is an ant, Bill Clinton knows x". *That's why we need the parentheses, so that we know what it is that's being said of x.* In the text book he writes it without parentheses sometimes, in which case it applies to just the thing that's right in front of the quantifier and variable. I'll do this too, sometimes, and you can, but don't get confused by doing it.

We're not going to follow a quantifier with a constant, because it leads to weird results. For example, if we wrote $\forall b(Ab)$ it would read "For all Binky the Clown, Binky the Clown is an ant", but this isn't good because Binky the Clown is an individual. The same goes for $\exists$. So **quantifiers should always be followed by a variable**. Before practicising translations from English into quantified logic, it's useful to practice reading quantified logic sentences in English, to get an idea of what they're like, because when translating you'll usually have to paraphrase the sentences into this kind of sentence first. Using the previous key:

$\forall x(Bbx \,\&\, Ax)$ reads as "For all x, Binky the Clown bartered with x and x is an ant".

$\neg\exists y(Cy \vee Ac)$ reads as "There does not exist a y such that either Bill Clinton knows y or Charles Barkley is an ant".

We'll often come across some particular types of sentences, so we need to know how to translate them without difficulty. Here we'll introduce another notation:

If A is a predicate with one input, then "A's" will be used to refer to the things which make the predicate true. So, for example, with our previous key, C's would mean "things Bill Clinton knows", and A's would mean "ants". And the same goes for the singulars (e.g. "an A" means "an ant", and "a C" means "a thing Bill Clinton knows"). This is just for convenience in our pseudo-language that's a mix of English and quantifier logic. Then:

"All A's are B's" translates as $\forall \mathbf{x}(\mathbf{Ax} \rightarrow \mathbf{Bx})$.

"Some A is B" translates as $\exists \mathbf{x}(\mathbf{Ax} \,\&\, \mathbf{Bx})$ ("Some A's are B's" means the same).

Note that since & is symmetric, but $\rightarrow$ isn't, "Some A is B" is equivalent to "Some B is A", but "All A's are B's" is not equivalent to "All B's are A's". If you think about it for a while, you should be able to satisfy yourself that this is really so. Also, note that **& tends to go with $\exists$, while $\rightarrow$ tends to go with $\forall$.** <u>This is very important</u>. If you're translating something and you think of putting a $\rightarrow$ with a $\exists$ or a & with a $\forall$ you need to make absolutely sure that's what is actually being said.

From these, we can see what similar sentences with negatives in them would be like:

"Not all A's are B's" translates as $\neg\forall \mathbf{x}(\mathbf{Ax} \rightarrow \mathbf{Bx})$.

"No A's are B's" translates as $\neg\exists \mathbf{x}(\mathbf{Ax} \,\&\, \mathbf{Bx})$.

It's important to notice that the negation of "All A's are B's" is "Not all A's are B's", not "No A's are B's". Some others:

"All A's aren't B's" translates as $\forall \mathbf{x}(\mathbf{Ax} \rightarrow \neg\mathbf{Bx})$ (this sentence is ambiguous in English, sometimes interpreted as "Not all A's are B's", but since we can simply use that sentence in such cases, we'll only use "All A's aren't B's" as meaning that that anything which is an A is not a B).

"Some A's aren't B's" translates as $\exists \mathbf{x}(\mathbf{Ax} \,\&\, \neg\mathbf{Bx})$.

In fact, some of these are equivalent to each other. Namely, to say that not all A's are B's is the same as to say that some A's aren't B's. And to say that no A's are B's is to say that all A's are not B's. Therefore:

$\neg\forall$**x**(**Ax** $\rightarrow$ **Bx**) **is logically equivalent to** $\exists$**x**(**Ax** & $\neg$**Bx**).

Remember that $Ax \rightarrow Bx$ is false if and only if Ax is true and Bx is false, so that $Ax \,\&\, \neg Bx$ is equivalent to $\neg(Ax \rightarrow Bx)$. Then we can see that:

$\neg\forall$**x**(**Ax** $\rightarrow$ **Bx**) **is logically equivalent to** $\exists$**x**$\neg$(**Ax** $\rightarrow$ **Bx**).

More generally, if $\mathcal{A}$ is any sentence (this time we'll have to use different letters to distinguish them from normal capital letters, which denote predicates) in quantifier logic, then:

$\neg\forall$**x**($\mathcal{A}$) **is logically equivalent to** $\exists$**x**$\neg$($\mathcal{A}$), **and likewise:**

$\neg\exists$**x**($\mathcal{A}$) **is logically equivalent to** $\forall$**x**$\neg$($\mathcal{A}$).

One last (for the moment) thing we need to know is the Universe of Discourse (UD). This is simply the set of things we're talking about. We could just say the UD is everything, and sometimes we might have reasons to do that, but we want to be able to limit ourselves. So, for example, if we say the UD is "Pitt students", then the sentence $\forall x(Ax)$ would mean "All Pitt students are ants". If we say the UD is Bill Clinton, Hillary Clinton, and Chelsea Clinton, then the sentence $\exists x(Bxc)$ would mean that one of these three bartered with Charles Barkley. In general, if we don't specific the UD, we'll assume it's everything. Actually, making the UD be everything leads to a logical paradox known as *Russell's Paradox*, so it's not really a good idea. But we don't need to worry about it for the purpose of this course.

Exercises

Translate the following sentences using this key:

UD: all animals

a: Alice

b: Betty

c: Chris

Ax: x is an ant
Bx: x is a bear
Rx: x is red
Lxy: x is larger than y.
Exy: x could eat y.

1. No bear could eat Alice.

2. Betty is larger than every ant.

3. Some animals aren't ants.

UD: all animals

a: Alice

b: Betty

c: Chris

Ax: x is an ant
Bx: x is a bear
Rx: x is red
Lxy: x is larger than y.
Exy: x could eat y.

4. Not every bear is red.

5. Not every animal is a red bear.

6. Some ants are red.

7. Chris is an ant which every bear could eat.

8. Every bear which is larger than Chris could eat Betty.

9. Some red ants are larger than Alice.

10. Some red animal could eat Chris.

11. Not every red animal is an ant.

12. There are no red bears.

13. Every ant is either red or a bear.

UD: all animals

a: Alice

b: Betty

c: Chris

Ax: x is an ant
Bx: x is a bear
Rx: x is red
Lxy: x is larger than y.
Exy: x could eat y.

14. Chris is larger than every animal which he could eat.

15. No ant is larger than Chris, unless it's also a bear.

16. Every red bear is either an ant or larger than Betty.

17. No ants are larger than Chris, who is a red bear.

18. There are no red ants which could eat Betty.

19. Every ant which isn't red could eat Chris.

20. Every animal which is larger than Betty could be eaten by Chris.

Homework #6, Due June 7, 2010

A. Translate each of the following sentences in the space provided, using the key given. (2 points each)

UD: all persons	Cx: x is a carpenter
k: Kelly	Px: x is a pilot
n: Nora	Rx: x is red
j: Jim	Txy: x is taller than y
	Lxy: x likes y

1. If Kelly is a carpenter, every pilot is red.

2. Nora is shorter than both Jim and Kelly.

3. Kelly likes every person who likes her.

4. No red people are pilots.

5. Some red carpenter likes Jim.

6. Not every carpenter is taller than Kelly.

7. Nobody who isn't red likes themselves.

8. Kelly is taller than everyone who Nora likes.

9. Jim doesn't like anyone who likes Nora.

UD: all persons

k: Kelly

n: Nora

j: Jim

Cx: x is a carpenter
Px: x is a pilot
Rx: x is red
Txy: x is taller than y
Lxy: x likes y

10. Nora and Jim like every carpenter.

11. Some pilot isn't also a carpenter

12. There are no pilots taller than Kelly.

13. Everyone who's shorter than Jim is either red or a pilot.

14. Every person liked either Kelly or Nora.

15. Some pilot doesn't like Kelly but does like Jim.

B. For each of the following, state whether it's true or false. If it's true, explain <u>in detail</u> how you can be sure of that. If it's false, give a counterexample (i.e. an example which shows it's false). (2 pts. each)

1. If $\{A, B\}$ is consistent, then $A \lor B$ is a tautology.
2. If $A \therefore B$ is a valid argument, then A and B are logically equivalent.
3. If $A \to B$ is contingent, B isn't a contradiction.
4. If $A \leftrightarrow B$ is a tautology, so is $A \to B$.
5. If $A \to B$ is a contradiction, so is $A \leftrightarrow B$.