Philosophy 500 — May 12th: Sentential Logic

Motivation

Goal: We want to translate sentences and arguments from ordinary language to a more formalized language in a way that would bring to the forefront their logical structure.

For example, the following two arguments have the same logical form:

If you lie you'll go to hell.
You won't go to hell.
$\therefore$ You don't lie.

If it's snowing, it's below freezing.
It isn't below freezing.
$\therefore$ It isn't snowing.

Though they talk about different topics, if we abstract away from the content they are essentially the same. Translating arguments into a formal language will allow to look at their formal structure without being distracted by the content, in order to determine whether they are valid or not. If they are, we can then consider the content to determine whether they're sound.

However, *for this to work, it's essential that we preserve the logical structure.*

For example:

Good translation:

Key:
A: You lie.
B: You'll go to hell.

Argument:
If A, then B
Not B
$\therefore$ Not A

Bad translation:

Key:
A: If you lie, you'll go to hell.
B: You won't go to hell.
C: You don't lie.

Argument:
A
B
$\therefore C$

Both of these are allowed, but the second isn't useful because it misses the logical structure of the argument. Note that the original argument is valid, even though you can translate it into an invalid argument in this way. So:

If an argument in ordinary language can translated to a valid argument in the formal language, it's valid.

But if it's translated into an invalid argument, this could be because the translation lost the logical structure, not necessary because the original argument wasn't valid.

Logical symbols

The formalized argument above still uses English words such as "if", "then", etc. We want to have a completely formal language, so we'll introduce more symbols:

Symbol	Name	Usage	"English" equivalents
$\neg$	Not/negation	$\neg A$	Not A
&	And/conjunction	$A \,\&\, B$	A and B / Both A and B
$\vee$	Or/disjunction	$A \vee B$	A or B / Either A or B
$\rightarrow$	Then/conditional	$A \rightarrow$B	If A then B / A implies B
$\leftrightarrow$	If and only if/biconditional	$A \leftrightarrow$B	A if and only if B / A iff B

The two arguments above can now be written as:

Key:
A: You lie.
B: You'll go to hell.

Argument:
$A \rightarrow B$
$\neg B$
$\therefore \neg A$

Key:
A: It's snowing.
B: It's below freezing.

Argument:
$A \rightarrow B$
$\neg B$
$\therefore \neg A$

This makes it clear that, as far as logical structure, the two arguments are identical.

Paraphrasing

We will often need to do some paraphrasing in our translations. For example:

1. "Alice and Bob both like ice-cream. Therefore, Alice likes ice-cream."

We can use A for "Alice likes ice-cream" and B for "Bob likes ice-cream". But we have to paraphrase "Alice and Bob both like ice-cream" as "Alice likes ice-cream and bob likes ice-cream". The argument then becomes: $A \,\&\, B \therefore A$

2. "If Tim is present, he's wearing a hat. Tim isn't wearing a hat. Therefore, Tim is absent."

We can use P for "Tim is present" and H for "Tim is wearing a hat". But before we can translate this, we need to paraphrase "Tim is absent" as "Tim isn't present", or even "It's not the case that Tim is present". We then get: $P \rightarrow H, \neg H \therefore \neg P$

3. We have to be careful. For example, some things which superficially look like negations aren't: to say that Bob is unhappy isn't the same as to say that Bob isn't happy, since a person might also be neither happy nor unhappy.

4. When creating translation keys, you should try to make letters correspond to simple sentences. For example, instead of letting A stand for "Alex isn't Russian" you should let it stand for "Alex is Russian", and then translate the original sentence as $\neg A$.

Additional usages

We also have other ways to express the same things in English. Here is a convenient table to help you along with this:

Either A or B	*means*	$A \vee B$
A unless B	*means*	$A \vee B$
A and B	*means*	$A \,\&\, B$
A but B	*means*	$A \,\&\, B$
If A, B	*means*	$A \rightarrow B$
A only if B	*means*	$A \rightarrow B$
A if B	*means*	$B \rightarrow A$
A whenever B	*means*	$B \rightarrow A$
A if and only if B	*means*	$A \leftrightarrow B$
Neither A nor B	*means*	$\neg A \,\&\, \neg$ B

Truth conditions / Meaning

Some of these have very clear meanings. For example, $\neg A$ is going to be true if A is false and false if A is true. Likewise, $A \,\&\, B$ will be true if both A and B are true, and false otherwise. But for $\vee$ it's not as simple, because the word "or" is used ambiguously in ordinarily language. To see this, compare these two examples:

"Anyone going to med school is either crazy or really into medicine."

"Either you'll raise your hands or I'll shoot you."

You wouldn't think the first one is false just because there are people in med school who are both crazy and really into it. On the other hand, the second is understood as meaning that if you raise your hands I won't shoot you. We express this difference by saying that in the first sentence it's an *inclusive or*, whereas in the second sentence it's an *exclusive or*.

Since we want to avoid ambiguity, in this class we will always treat or's as inclusive. Therefore, $A \vee B$ will be true so long as at least one of them is true. In other words, $A \vee B$ is true except if both A and B are false.

Finally, we will treat conditionals as *material conditionals*. This is just a technical term to mean that $A \rightarrow B$ *is true if* A *is false or if* A *and* B *are both true, and is false if* A *is true and* B *is false.*

For example: "If it's raining, it's cloudy" will come out as true this way, which is what we wanted.

To summarize:

$\neg A$	*is true if and only if*	A is false
$A \,\&\, B$	*is true if and only if*	Both A and B are true
$A \vee B$	*is true if and only if*	At least one of A and B is true
$A \vee B$	*is true if and only if*	A and B aren't both false
$A \rightarrow B$	*is true if and only if*	A is false or both A and B are true
$A \rightarrow B$	*is true if and only if*	It's not the case that: A is true and B is false
$A \leftrightarrow B$	*is true if and only if*	A and B are both true or A and B are both false

Complicated sentences and parentheses

We are now in a position to translate many types of sentences. However, in order to prevent ambiguity we will need to use parentheses.
For example:

$\neg(A \,\&\, B)$ means it's not the case that A and B are both true.

$(\neg A) \,\&\, B$ means that A is false and B is true.

Remember that in math there is a certain order in which operations are done when there aren't parentheses to say otherwise, so that $2 + 3 \times 4$ means $2 + (3 \times 4)$, not $(2+3)\times 4$. We express this by saying that multiplication is done before addition. Likewise, *in logic we do negation before the binary operators.*

Therefore: $\neg A \& B$ means the same as $(\neg A) \& B$, and we can leave out the parentheses.

On the other hand, there is no order of priority between the various binary operators. So if you were to write $A \& B \vee C$, it would be ambiguous: does it mean $(A \& B) \vee C$ or does it mean $A \,\&\, (B \vee C)$? These two don't mean the same thing, so without the parentheses we simply don't know how to interpret it, and so we don't consider it a proper sentence.

So you must always make sure that anything you write down isn't ambiguous due to missing parentheses.

Even if we're dealing with the same operator it might be ambiguous: $A \to B \to C$ could mean either $A \to (B \to C)$ or $(A \to B) \to C$. These don't mean the same thing, so without parentheses it's meaningless.

The only cases where parentheses aren't necessary is when you have several letters joined by &'s or several letters joined by ∨'s. For example: $A \,\&\, (B \,\&\, C)$ and $(A \,\&\, B) \,\&\, C$ are logically equivalent, so we can simply write $A \,\&\, B \,\&\, C$. But this only works for several &'s and for several ∨'s (but not when there's some of each, as above).

If you're unsure whether they're needed, it's always better to include parentheses.

E.g. #1. Using the key given, translate each of the following sentences.

A: Alice voted for Lincoln.
B: Bob voted for Lincoln.
E: Lincoln got elected.
P: Lincoln became president.

(a) Lincoln got elected even though neither Alice nor Bob voted for him.

(b) Lincoln only became president if Alice voted for him.

(c) Unless Lincoln got elected or Alice voted for him, he didn't become president.

(d) Alice voted for Lincoln only if he got elected and became president.

(e) Alice and Bob didn't both vote for Lincoln.

E.g. #2. Using the key given, translate each of the following sentence.

A: Alex is an astronaut.
B: Betty is an astronaut.
C: Cindy is an astronaut.
P: Pittsburgh is in Russia.
T: Toronto is in Russia.

(a) Toronto and Pittsburgh aren't both in Russia.

(b) If Alex and Betty are both astronauts, so is Cindy.

(c) Toronto is in Russia unless neither Cindy nor Alex is an astronaut.

(d) Toronto is in Russia if and only if it isn't the case that both Betty and Cindy are astronauts.

(e) Alex is an astronaut only if Pittsburgh and Toronto are both in Russia.

(f) If Alex isn't an astronaut, neither is Betty, but Cindy is an astronaut either way.

(g) Unless Toronto is in Russia, Cindy is an astronaut but Alex isn't.

(h) If any out of Alex, Betty, and Cindy is an astronaut, Pittsburgh is in Russia.

(i) If Alex is an astronaut and so is Betty, Toronto and Pittsburgh aren't both in Russia.

(j) Alex and Betty are both astronauts if and only if neither of them is.

(k) If Betty is an astronaut, then Toronto isn't in Russia unless Pittsburgh is.

(l) Either Toronto is in Russia or Pittsburgh isn't.

(m) Unless Alex is an astronaut, neither Betty nor Cindy is.

Relating logical concepts

For each of the follow, state whether it's true or false. If it's true, explain why. If it's false, give an example which shows that.

1. If A is a tautology, then $A \,\&\, B$ is a tautology, whatever B is.

2. If $A \therefore B$ is a valid argument, then $A \to B$ is a tautology.

3. If A is a tautology, then the argument $B \therefore A$ is valid, whatever B is.

4. If A is a tautology, and B is contingent, then the argument $A \therefore B$ is invalid.

5. If $A \,\&\, B$ is contingent, then both A and B are contingent.

6. If $A \leftrightarrow B$ is a tautology, then the set $\{A, B\}$ is consistent.

Homework #2, Due May 17, 2010

Please answer sections A and B in the spaces provided and section C on another piece of paper, and staple them together. Thanks!

A. Using the key given, translate each of the following sentences. (1 pt. each)

A: Ashley is a turtle.
J: James is a turtle.
M: Melissa is a turtle.
S: Sidney is a crocodile.
V: Vern is an crocodile.

1. Neither Ashley nor Melissa is a turtle, but James is.

2. Unless Melissa and James are both turtles, Vern isn't a crocodile.

3. Vern isn't a crocodile if both Ashley and James are turtles.

4. Sidney is a crocodile if and only if Vern isn't one.

5. Either James is a turtle, or Melissa is only a turtle if Ashley is one too.

6. If Ashley isn't a turtle, neither is James. But Vern is a crocodile regardlessly.

7. Unless Sidney is a crocodile, neither Vern nor Sidney is a crocodile.

8. Either James or Ashley is a turtle, but they're not both turtles.

9. Melissa is a turtle if and only if neither James and Ashley is one.

10. If Melissa is a turtle and James isn't one, then Vern isn't a crocodile.

B. Which of the following sets are consistent? If you want, you can think in terms of the key above, though it shouldn't make a difference to the answers whether or not you do. (Write 'C' for consistent, 'I' for inconsistent, 1 pt. each)

___ 1. $\{A \,\&\, J, A \,\&\, \neg J, S \rightarrow A\}$

___ 2. $\{J, J \rightarrow V, \neg V\}$

___ 3. $\{A \,\&\, J, A \rightarrow J\}$

___ 4. $\{\neg A, A \rightarrow J, J\}$

C. For each of the following, state whether it's true or false. If it's true, explain in detail how you can be sure of that. If it's false, give a counterexample (i.e. an example which shows it's false). (2 pts. each)

1. If a set of sentences is inconsistent, one of them is a contradiction.

2. If all of an argument's premises are tautologies, but its conclusion isn't, the argument is invalid.

3. If $\{A, B, C\}$ is a consistent set, then the argument $A, B \therefore C$ is valid.

4. If A is a tautology, so is $A \vee B$, whatever B is.

5. If A and B are both contingent, then $A \rightarrow B$ is a tautology.

6. If $\neg A$ is contingent, so is A.

7. If an argument has one false premise and one true premise, it's invalid.

8. If an argument is sound and its conclusion is a tautology, its premises are also tautologies.