Logarithms: change of base and logarithmic equations

Mr. Neeman, 10B, Sept. 7, 2011.

Change of base

Suppose $\log_a b = y$. Then $a^y = b$
If c is a positive number that's not 1, we can take logarithm base c on both sides:
$\log_c(a^y) = \log_c b$
$y \log_c a = \log_c b$
Therefore, $y = \dfrac{\log_c b}{\log_c a}$
So we get:

$$\boxed{\log_a b = \frac{\log_c b}{\log_c a}}$$

We can use this in solving equations and manipulating expressions.

E.g. #1. Express $\log_{25} 20$ in terms of a logarithm with base 5.
$\log_{25} 20 = \dfrac{\log_5 20}{\log_5 25} = \dfrac{\log_5 20}{2}$

E.g. #2. Express $\log_{\frac{1}{4}} x$ in terms of $\log_2 x$.
$\log_{\frac{1}{4}} x$
$= \dfrac{\log_2 x}{\log_2 \frac{1}{4}}$
$= \dfrac{\log_2 x}{\log_2(2^{-2})}$
$= -\dfrac{1}{2} \log_2 x$

E.g. #3. Express $\log_4(x^2) - \log_2 y$ as a single logarithm.
First, we get these into the same base. We could use base 2 or base 4, let's try base 2.
We know $\log_4(x^2) = \dfrac{\log_2(x^2)}{\log_2 4} = \dfrac{\log_2(x^2)}{2} = \log_2 x$
So $\log_4(x^2) - \log_2 y = \log_2 x - \log_2 y = \log_2 \dfrac{x}{y}$

E.g. #4. Simplify $\log_4(x^3) - 3\log_2 \sqrt{x}$
$\log_4(x^3) - 3\log_2 \sqrt{x}$
$= \dfrac{\log_2(x^3)}{\log_2 4} - 3\log_2 \sqrt{x}$
$= \dfrac{\log_2(x^3)}{2} - 3\log_2 \sqrt{x}$
$= \log_2(\sqrt{x^3}) - \log_2(\sqrt{x}^3)$
$= 0$

Solving equations

We can use the logarithm laws and the change of base formula to solve logarithmic equations which we wouldn't have been able to solve otherwise. Generally speaking, we will try to manipulate the equation to get a single logarithm on one side and a constant on the other side (though sometimes it works to have one logarithm on each side). We can then apply the exponential function to both sides of the equation and solve from there. However, we have to remember that logarithms can only be applied to positive numbers, so we have to check for each solution that it doesn't give us something undefined. This is similar to solving rational equations, where we have to make sure that our solutions don't involve a division by zero.

Note: As usual when given an equation to solve, you must keep in mind that it might not have any solutions, in which your answer should be that there are no solutions. This is like with previous types of equations (e.g. if I ask you to solve $x^2 + 4 = 0$).

E.g. #1. Solve $\log_2(3x+1) = 1 + \log_2(2x-3)$
$\log_2(3x+1) - \log_2(2x-3) = 1$
$\log_2 \dfrac{3x+1}{2x-3} = 1$
$\dfrac{3x+1}{2x-3} = 2$
$3x+1 = 4x-6$
$7 = x$
We then check that when $x = 7$, $3x+1$ and $2x-3$ are both positive, so we don't get any logarithms being undefined. We can also subtitute back into the original equation to check our answer. In the end, we have $x = 7$.

E.g. #2. Solve $\log_3(2x-3) + \log_3(x-3) = \log_3(x+1)$
$\log_3((2x-3)(x-3)) = \log_3(x+1)$
$\log_3((2x-3)(x-3)) - \log_3(x+1) = 0$
$\log_3 \dfrac{(2x-3)(x-3)}{x+1} = 0$
$\dfrac{(2x-3)(x-3)}{x+1} = 1$
$(2x-3)(x-3) = x+1$
$2x^2 - 9x + 9 = x + 1$
$2x^2 - 10x + 8 = 0$
$x^2 - 5x + 4 = 0$
$(x-4)(x-1) = 0$
Therefore, $x = 1$ or $x = 4$. However, $2x-3$, $x-3$, and $x+1$ must all be positive. With $x = 4$, this is the case. But with $x = 1$, the first two are negative, so this can't be a solution. So our only solution is $x = 4$.

E.g. #3. $\log_2 x = \log_4 x$
$\log_2 x = \dfrac{\log_2 x}{\log_2 4}$
$\log_2 x = \dfrac{\log_2 x}{2}$
$\dfrac{1}{2}\log_2 x = 0$
$\log_2 x = 0$
$x = 1$

E.g. #4. $\log_3 x = \log_9(2x^3)$

$\dfrac{\log_9 x}{\log_9 3} = \log_9(2x^3)$

$\dfrac{\log_9 x}{\log_9(9^{\frac{1}{2}})} = \log_9(2x^3)$

$\dfrac{\log_9 x}{\frac{1}{2}} = \log_9(2x^3)$

$2\log_9 x = \log_9(2x^3)$

$\log_9(x^2) = \log_9(2x^3)$

$x^2 = 2x^3$

$x^2(1-2x) = 0$

So $x = 0$ or $x = \dfrac{1}{2}$. But $x = 0$ can't be a solution, because you can't take logarithm of zero. So $x = \dfrac{1}{2}$.

<u>Homework (for Monday, Sept. 12)</u>

#H1. Express $\log_9 x$ in terms of $\log_3 x$.

#H2. Express $\log_{\frac{1}{5}} 7$ in terms of a logarithm with base 5.

#H3. Express $\log_3(x^4) - \log_9(y^2)$ as a single logarithm.

#H4. Simplify $\dfrac{\log_4 20}{\log_2 20}$.

#H5. Solve $\log_2 x + \log_2(x-2) = 3$.

#H6. Solve $\log_2(5x) - \log_2(x-3) = 4$.

#H7. Solve $\log_2 x + 2 = \log_2(2x+4)$.

#H8. Solve $\log_2(x+2) = \log_2(2x+4)$ (note this is not the same as #H7).

#H9. Solve $2\log_2(x+1) - \log_2(x^2-1) = \ln\left(\dfrac{1}{e}\right)$.

#H10. Solve $\log_2(x-2) = \log_4(x^3-8)$.

#H11. Solve $\log_{10}(x+3) - \log_{10}(x-1) = \log_{10} x$.

#H12. Solve $3\log_5(x-3) = 6$.

#H13. Solve $\log_3(x+2) - \log_3(x-2) = 4$.

#H14. Express $\log_{\frac{1}{\sqrt{2}}}(x^3) - \log_2\sqrt{x}$ in terms of $\log_2 x$.

#H15. Express $\log_3(xy^{-2}) - \log_3(x^2y) + \dfrac{1}{2}\log_9 x$ as a single logarithm.

<u>Selected homework solutions</u>

These will be posted on Friday.