Logarithm laws

Mr. Neeman, 10B, Sept. 6, 2011.

Using the fact that $y = \log_a x$ if and only if $a^y = x$, we can find logarithm laws which are analogous to some of our power laws.
For the following, suppose $a^y = x$ and $a^z = w$. Thus, $y = \log_a x$ and $z = \log_a w$. As usual, we need $a > 0$, $a \neq 1$, $x > 0$ and $w > 0$.

#1. Power law: $a^y a^z = a^{y+z}$
$xw = a^{\log_a x + \log_a w}$
Taking logarithm base a of both sides, we get:
$\log_a(xw) = \log_a x + \log_a w$

#2. Power law: $\dfrac{a^y}{a^z} = a^{y-z}$
$\dfrac{x}{w} = a^{\log_a x - \log_a w}$
Taking logarithm base a of both sides, we get:
$\log_a \dfrac{x}{w} = \log_a x - \log_a w$

#3. Power law: $(a^y)^n = a^{yn}$
$x^n = a^{n \log_a x}$
Taking logarithm base a of both sides, we get:
$\log_a(x^n) = n \log_a x$

#4. Power law: $\sqrt[n]{a^y} = a^{\frac{y}{n}}$
$\sqrt[n]{x} = a^{\frac{\log_a x}{n}}$
Taking logarithm base a of both sides, we get:
$\log_a(\sqrt[n]{x}) = \dfrac{\log_a x}{n}$

#5. Power law: $a^0 = 1$
Taking logarithm base a of both sides, we get:
$0 = \log_a 1$
So: $\log_a 1 = 0$

#6. $a^1 = a$
Taking logarithm base a of both sides, we get:
$1 = \log_a a$
So: $\log_a a = 1$

To sum up, and remembering that the logarithmic and exponential functions are inverses of each other:

1. $\log_a x + \log_a y = \log_a xy$
2. $\log_a x - \log_a y = \log_a \dfrac{x}{y}$
3. $n \log_a x = \log_a(x^n)$
4. $\dfrac{\log_a x}{n} = \log_a \sqrt[n]{x}$
5. $\log_a 1 = 0$
6. $\log_a a = 1$
7. $\log_a(a^x) = x$
8. $a^{\log_a x} = x$

These laws can be used to simplify and manipulate expressions, solve equations, etc. We will come back to solving equations, for now we'll focus on simplifying and manipulating expressions.

<u>Examples</u>

#1. Simplify $\log_3(x^3) - \log_3 x$

$\log_3(x^3) - \log_3 x$
$= 3\log_3 x - \log_3 x = 2\log_3 x$

#2. Express $\log_2 \dfrac{x^2}{z^{-1}\sqrt{y}}$ in terms of simpler logarithms (logarithms of x, of y, and of z).

$\log_2 \dfrac{x^2}{z^{-1}\sqrt{y}}$
$= \log_2(x^2) - \log_2(z^{-1}\sqrt{y})$
$= \log_2(x^2) - \log_2(z^{-1}) - \log_2\sqrt{y}$
$= 2\log_2 x + \log_2 z - \dfrac{1}{2}\log_2 y$

#3. Simplify $\log_{\frac{1}{2}}(x^2+2x+1) - \log_{\frac{1}{2}}(x+1)$

$\log_{\frac{1}{2}}(x^2+2x+1) - \log_{\frac{1}{2}}(x+1)$
$= \log_{\frac{1}{2}}((x+1)^2) - \log_{\frac{1}{2}}(x+1)$
$= \log_{\frac{1}{2}} \dfrac{(x+1)^2}{x+1}$
$= \log_{\frac{1}{2}}(x+1)$

#4. Express $3\log_4(x^2) - \log_4\sqrt{y} - \dfrac{1}{2}\log_4 z$ as one logarithm.

$3\log_4(x^2) - \log_4\sqrt{y} - \dfrac{1}{2}\log_4 z$
$= \log_4((x^2)^3) - \log_4\sqrt{y} - \log_4\sqrt{z}$
$= \log_4 \dfrac{x^6}{\sqrt{yz}}$

#5. Simplify $\dfrac{\log_2(x^3)}{\log_2(8y^3z^3)}$

$\dfrac{\log_2(x^3)}{\log_2(8y^3z^3)}$
$= \dfrac{\log_2(x^3)}{\log_2((2yz)^3)}$
$= \dfrac{\log_2(x^3)}{\log_2((2yz)^3)}$
$= \dfrac{3\log_2 x}{3\log_2(2yz)}$
$= \dfrac{\log_2 x}{\log_2(2yz)}$
$= \dfrac{\log_2 x}{\log_2 2 + \log_2(yz)}$
$= \dfrac{\log_2 x}{1 + \log_2(yz)}$

And, if you think it's simpler:
$= \dfrac{\log_2 x}{1 + \log_2 y + \log_2 z}$

#6. Simplify $\log_5(\sqrt{29} - 2) + \log_5(\sqrt{29} + 2)$
$\log_5(\sqrt{29} - 2) + \log_5(\sqrt{29} + 2)$
$= \log_5((\sqrt{29} - 2)(\sqrt{29} + 2))$
$= \log_5(29 - 4)$
$= \log_5 25$
$= 2$

<u>Homework (for Thursday)</u>

#H1. Express each of the following in terms of simpler logarithms and simplify if possible:

(a) $\log_3 \dfrac{x^3}{9}$
(b) $\log_2(\sqrt[4]{z}xy^2)$
(c) $\ln \dfrac{e^2x^3}{y}$
(d) $\log_{\frac{1}{2}} \dfrac{\sqrt[3]{x}}{4y^3}$
(e) $\log_{10}(xy^{-2})$
(f) $\log_5 \left(\left(\dfrac{y}{x^3}\right)^{-2}\right)$

#H2. Express each of the following as one logarithm, without a coefficient in front of it:

(a) $\dfrac{\log_2 3x}{2} - 2\log_2 y$
(b) $\log_{\frac{1}{3}}(y^2) - 3\log_{\frac{1}{3}} \sqrt{y}$
(c) $2\log_3(xy^2) + \log_3 \dfrac{x}{y^3}$
(d) $-\dfrac{\log_4 \sqrt{y}}{2} + \log_4 y - 2\log_4 x$
(e) $3\ln(x^2) - \dfrac{\ln((xy^2)^4)}{2}$

#H3. Simplify each of the following as far as possible:

(a) $\log_3 1 + \log_3(x^3 - 1) - \log_3(x + 1)$
(b) $\dfrac{\log_2(x^2)}{\log_2(y^4)}$
(c) $\ln(\sqrt{x} + 4) + \ln(\sqrt{x} - 4)$
(d) $\ln(e^3xy) - \ln(e^2y^2)$

Solutions to selected questions

#H1. (c) $\ln \dfrac{e^2x^3}{y}$
$= \ln(e^2x^3) - \ln y$
$= \ln(e^2) + \ln(x^3) - \ln y$
$= 2 + 3\ln x - \ln y$

#H2. (a) $\dfrac{\log_2 3x}{2} - 2\log_2 y$
$= \log_2 \sqrt{3x} - \log_2(y^2)$
$= \log_2 \dfrac{\sqrt{3x}}{y^2}$

#H3. (d) $\ln(e^3xy) - \ln(e^2y^2)$
$= \ln \dfrac{e^3xy}{e^2y^2}$
$= \ln(exy^{-1})$
$= \ln e + \ln \dfrac{x}{y}$
$= 1 + \ln \dfrac{x}{y}$
Or, if you think it's simpler:
$= 1 + \ln x - \ln y$