Exponential equations

Mr. Neeman. 10B, August 18th, 2011

Exponential equations are equations which involve exponential functions (e.g. 2^x or composite functions with exponentials (e.g. $2^(x^2)$ which is a composition of an exponential and a quadratic function).

The simplest type of exponential equation

The simplest exponential equations are like this one:
$2^x = 2^4$
Here, we use the fact that the exponential function is injective to conclude that the two exponents must be equal: $x = 4$.
To solve such equations, it's very important to be proficient with the power laws so you can recognize powers of the base.
E.g. $4^x = \frac{1}{8}$
$4^x = 4^{-\frac{3}{2}}$
Therefore, $x = -\frac{3}{2}$

More complicated exponents

Sometimes, the exponents will be more complicated than just x, or both exponents will involve x instead of just one of them. In such cases we proceed the same way, but have more work to do.
E.g. $2^x = 4^{x^2}$
$2^x = 2^{2x^2}$
Therefore $x = 2x^2$, which is a quadratic equation so we know how to solve it:
$2x^2 - x = 0$
$x(2x - 1) = 0$, so our solutions are $x = 0$ and $x = \frac{1}{2}$.

e.g. $\frac{(9)^x}{3} = 9^{-x}$
$3^{2x-1} = 3^{-x}$
Therefore, $2x - 1 = -x$, so $x = \frac{1}{3}$

Substitution

In other cases, we can't get the equation into the form of having the same number raised to some power on each side. For example:
$-2^x + 6 = 4^x$
We can use the power laws to get the same base in both:
$-2^x + 6 = 2^{2x}$
But now we're stuck. What we need to do is use a substitution: $y = 2^x$, and express everything in terms of 2^x:
$-2^x + 6 = (2^x)^2$
$-y + 6 = y^2$

We can then solve for y:
$y^2 + y - 6 = 0$
$(y+3)(y-2) = 0$
So the solutions are $y = -3$ and $y = 2$. Now, if $y = -3$, that means $2^x = -3$. But this has no solutions, because 2^x is always positive (recall its range is $]0, \infty[$). On the other hand, if $y = 2$, we get $2^x = 2$, which has solution $x = 1$. Therefore, our answer is $x = 1$

E.g. $5^{2x+1} = 2 - 9(5^x)$
Here we've got three terms, so we won't be able to get it down to just two powers of 5. So we use substitution: $y = 5^x$.
$5(5)^{2x} = 2 - 9(5^x)$
$5(5^x)^2 = 2 - 9(5^x)$
$5y^2 = 2 - 9y$
$5y^2 + 9y - 2 = 0$
$(5y-1)(y+2) = 0$
Therefore, $y = \frac{1}{5}$, in which case $x = -1$, or $y = -2$, which doesn't have any solutions for x (since 5^x is always positive).

<u>Exercises</u>
The A exercises are on the simplest type of exponential equations, the B exercises involve complicated exponents, and the C type involve substitution

#A.1. $3^x = \frac{1}{27}$

#A.2. $5(2^{2x}) = 20$

#A.3. $5^x - 5 = -4$

#B.1. $\sqrt{2^x} = 4^{2x-2}$

#B.2. $9^{(\frac{x^2}{2}+2)} = 3^{4x}$

#B.3. $\sqrt{5}^{-2x} = \sqrt[3]{5}^{x+3}$

#C.1. $4^{(x+\frac{1}{2})} + 2 = 5(2^x)$

#C.2. $-3^{1-x} = 3^x - 4$

#C.3. $\frac{1}{5^{2x}} = 5^x$

#C.4. $16^{2x} = 2(16^x) + 8$

<u>Solutions</u>

#A.1. $3^x = \frac{1}{27}$
$3^x = 3^{-3}$
Therefore, $x = 3$.

#A.2. $5(2^{2x}) = 20$
$2^{2x} = 4$
$(2^2)^x = 4$
$4^x = 4$
Therefore, $x = 1$.

#A.3. $5^x - 5 = -4$
$5^x = 1$
Therefore, $x = 0$.

#B.1. $\sqrt{2^x} = 4^{2x-2}$
$2^{\frac{x}{2}} = 4^{2x-2}$
$2^{\frac{x}{2}} = (2^2)^{2x-2}$
$2^{\frac{x}{2}} = 2^{4x-4}$
Therefore, $\dfrac{x}{2} = 4x - 4$, so that $x = 8x - 4$ and $x = \dfrac{4}{7}$

#B.2. $9^{(\frac{x^2}{2}+2)} = 3^{4x}$
$(3^2)^{(\frac{x^2}{2}+2)} = 3^{4x}$
$3^{x+4} = 3^{4x}$
Therefore, $x^2 + 4 = 4x$.
$x^2 - 4x + 4 = 0$
$(x-2)^2 = 0$
Therefore $x = 2$.

#B.3. $\sqrt{5}^{-2x} = \sqrt[3]{5}^{x+3}$
$5^{-x} = 5^{\frac{x+3}{3}}$
Therefore $-x = \dfrac{x+3}{3}$, so that $-3x = x + 3$ and $x = -\dfrac{3}{4}$

#C.1. $4^{(x+\frac{1}{2})} + 2 = 5(2^x)$
$(2^2)^{(x+\frac{1}{2})} + 2 = 5(2^x)$
$2^{2x+1} + 2 = 5(2^x)$
$2(2x)^2 + 2 = 5(2^x)$
We use the substitution $y = 2^x$.
$2y^2 + 2 = 5y$
$2y^2 - 5y + 2 = 0$
$(2y-1)(y-2) = 0$
Therefore the solutions for y are $y = \dfrac{1}{2}$ and $y = 2$. Now we find the corresponding values of x.
If $y = \dfrac{1}{2}$, that means $2^x = \dfrac{1}{2}$, so $2^x = 2^{-1}$ and $x = -1$.
If $y = 2$, that means $2^x = 2$, so $x = 1$.
Therefore, the solutions are $x = 1$ and $x = -1$.

#C.2. $-3^{1-x} = 3^x - 4$
$-\dfrac{3}{3^x} = 3^x - 4$
Use the substitution $y = 3^x$:

$-\dfrac{3}{y} = y - 4$, which is a fractional equation.
We note that $y \neq 0$, and multiply both sides by y:
$-3 = y^2 - 4y$
$0 = y^2 - 4y + 3$
$0 = (y-3)(y-1)$
Therefore, $y = 1$ or $y = 3$. Now we need to find the corresponding values of x.
If $y = 1$, then $3^x = 1$, so that $3^x = 3^0$ and $x = 0$.
If $y = 3$, then $3^x = 3$, so that $x = 1$.
Therefore, the solutions are $x = 0$ and $x = 1$.

#C.3. $\dfrac{1}{5^{2x}} = 5^x$
$\dfrac{1}{(5^x)^2} = 5^x$
Use the substitution $y = 5^x$:
$\dfrac{1}{y^2} = y$
We note that $y \neq 0$ and multiply both sides by y^2:
$1 = y^3$
Therefore, $y = 1$ (this is the only solution to this cubic equation, which can be seen by moving the 1 to the left and factorizing using the difference of cubes formula), which means $5^x = 1$, so $x = 0$.

#C.4. $16^{2x} = 2(16^x) + 8$
$(16^x)^2 = 2(16^x) + 8$
Use the substitution $y = 16^x$:
$y^2 = 2y + 8$
$y^2 - 2y - 8 = 0$
$(y-4)(y+2) = 0$
Therefore, $y = 4$ or $y = -2$. Now we need to find the corresponding values of x.
If $y = 4$, then $16^x = 4$, so that $16^x = 16^{\frac{1}{2}}$ and so $x = \dfrac{1}{2}$.
If $y = -2$, then $16^x = -2$. However, this has no solutions, since 16^x is always positive, whatever x is (the range of 16^x is $]0, \infty[$. So for this solution for y there is no corresponding value for x.
Therefore, the only solution is $x = \dfrac{1}{2}$.

<u>Practice exercises</u>

#P1. $\left(\dfrac{3}{5}\right)^{x+2} = \dfrac{27}{125}$

#P2. $4^{3x-1} = 8^{-2x+1}$

#P3. $49^x - 7^{x+1} - 7^x = -7$

#P4. $\dfrac{1}{3^{3x-1}} = 27$

#P5. $4^{2x} - 4^{x+2} = -16 - 4^x$

Solutions for practice exercises

#P1. $\left(\frac{3}{5}\right)^{x+2} = \frac{27}{125}$
$\left(\frac{3}{5}\right)^{x+2} = \left(\frac{3}{5}\right)^{3}$
$\left(\frac{3}{5}\right)^{x} = \frac{3}{5}$
Therefore, $x = 1$

#P2. $4^{3x-1} = 8^{-2x+1}$
$2^{2(3x-1)} = 2^{3(-2x+1)}$
$2^{6x-2} = 2^{-6x+3}$
Therefore, $6x - 2 = -6x + 3$, so that $12x = 5$ and $x = \frac{5}{12}$.

#P3. $49^x - 7^{x+1} - 7^x = -7$
$7^{2x} - 7(7^x) - 7^x = -7$
$(7^x)^2 - 8(7^x) + 7 = 0$
Substitute $y = 7^x$:
$y^2 - 8y + 7 = 0$
$(y-1)(y-7) = 0$
Therefore $y = 1$ or $y = 7$.
If $y = 1$, then $x = 0$.
If $y = 7$ then $x = 1$.
Therefore the solutions are $x = 0$ and $x = 1$.

#P4. $\frac{1}{3^{3x-1}} = 27$
$3^{1-3x} = 27$
$3^{1-3x} = 3^3$
Therefore, $1 - 3x = 3$, so that $x = -\frac{2}{3}$.

#P5. $4^{2x} - 4^{x+2} = -16 - 4^x$
$(4^x)^2 - 4^2 4^x = -16 - 4^x$
$(4^x)^2 - 15(4^x) + 16 = 0$
Substitute $y = 4^x$:
$y^2 - 15y + 16 = 0$
$(y-16)(y+1) = 0$
Therefore $y = 16$ or $y = -1$.
If $y = 16$, then $x = 2$.
If $y = -1$, we have no corresponding value of x, since 4^x is always positive.
Therefore, $x = 2$.

Homework (due Monday Aug. 21st)

Solve each of the following equations. This means finding all solutions, and doing so in a way which ensures that you haven't missed any solutions.

#H1. $\dfrac{2^{2x}}{4^{3x}} = \sqrt{8}^3$

#H2. $3^{2x} - \dfrac{2}{3^{-x}} = 3$

#H3. $\dfrac{3}{4^x} = 24$

#H4. $4^{-x} = 2^{2-4x}$

#H5. $4^{2x+\frac{1}{2}} = 3 - 5(4^x)$

#H6. $2^{x^2+2} = 4^{-x}$