Quadratic functions

Mr. Neeman. 10B, July 29, 2011

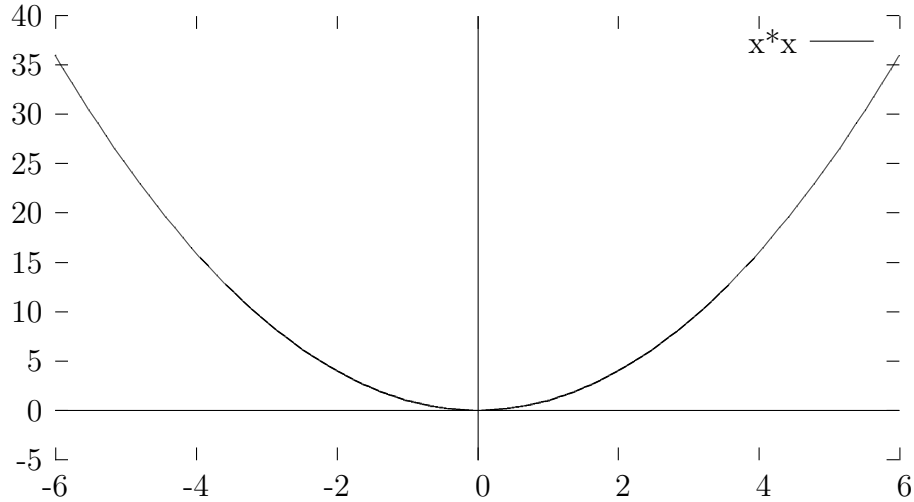
General quadratic functions

A quadratic function is anything of the form $f(x) = ax^2 + bx + c$, where a, b, c are constants. We will study the properties of such functions and their graphs, but we will start with the simplest ones and work our way up.

$f(x) = x^2$

The simplest case is $f(x) = x^2$. Its graph is a parabola, as show in in figure 1.

Figure 1

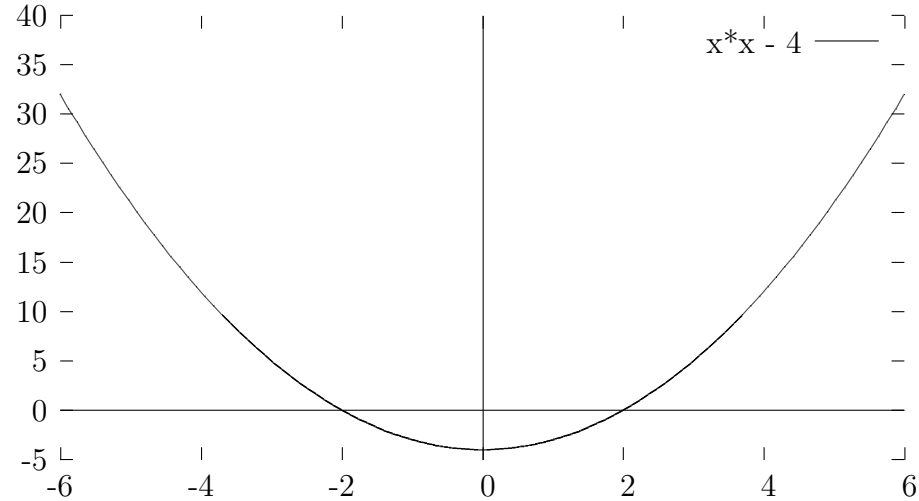


Using the graph and the function's definition, we can identify it properties:
1. The function is not injective, since each positive number has two preimages.
2. The function is not surjective, since each negative number has no preimage.
3. The function is increasing for positive x and decreasing for negative x.
4. The function is concave up everywhere. This just means that the way it's curving is always up, which is to say that its gradient is increasing as x increasing.
5. Its range is $[0,]\infty$
6. It's symmetric around the y-axis.
7. It intersects the axes at the origin.

$f(x) = x^2 + c$

Our first complication is to add a constant. What this does is it simply adds c to the function, so for every value of x our function will be c higher than before. Graphically, this means the curve is shifted up by c. If c is negative, that means it's shifted down, of course. Figure 2 shows an example with $c = -4$.

Figure 2



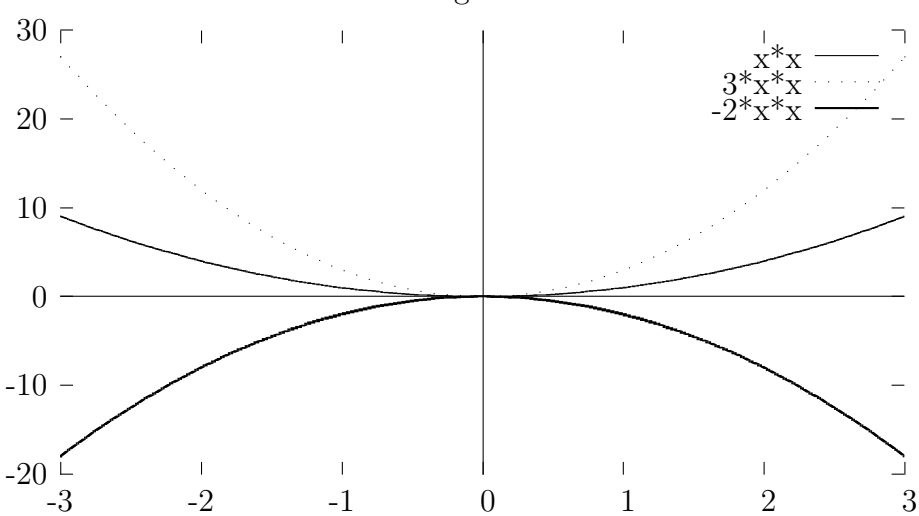
This function shares many of the properties we identified for x^2. However, there are some differences. For example, it doesn't intersect the origin. Instead, it intersects the x axis at $(-2, 0)$ and $(2, 0)$ and intersects the y axis at $(0, -4)$.

As an exercises, identify this function's properties using the graph and the mapping.

$\underline{f(x) = ax^2}$

Next, we look at $f(x) = ax^2$. This simply rescales the function vertically. Figure 3 shows some examples. Notice that if a is negative the parabola faces down instead of up. A way to think about it is that multiplying a function by -1 is to reflect it along the x axis.

Figure 3



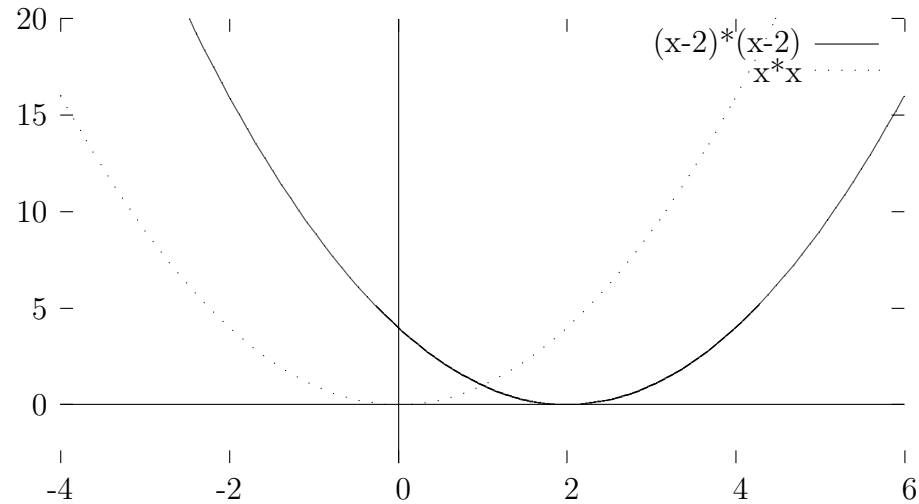
Again, for each of these identify its various properties.

$\underline{f(x) = (x - k)^2}$

Next, we look at $f(x) = (x - k)^2$, and the specific example $f(x) = (x - 2)^2$. But notice that multiplying it out gives $f(x) = x^2 - 4x + 4$, so that we're moving towards the general form.

What this function represents is a translation to the right of x^2. One way to think about it is using a substitution. Let's say $g(z) = z^2$. Then if we substitute $z = x - 2$, we'll see that $g(x-2) = (x-2)^2 = f(x)$. Now $z = 0$ when $x = 2$, so the z axis would be a vertical line at $x = 2$, and that's where $f(x)$ is centered. The graph is shown in figure 4.

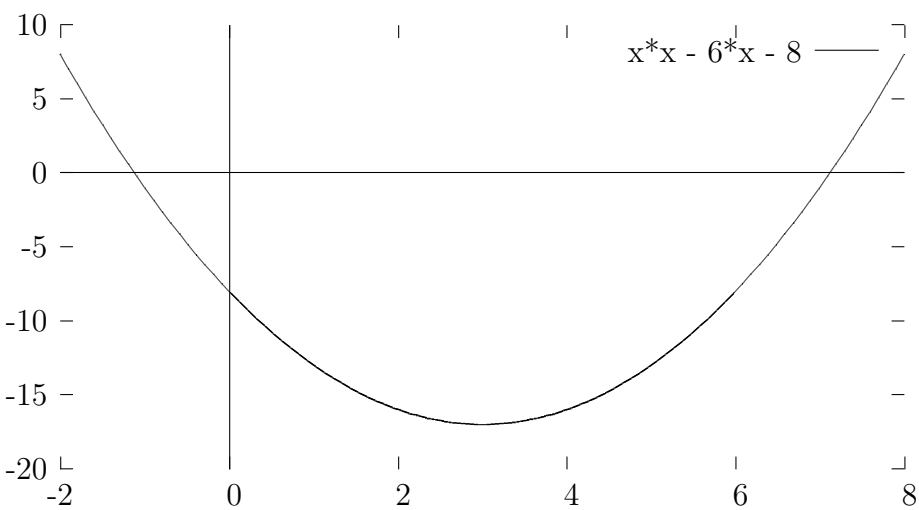
Figure 4



<u>Combination of transformations</u>

Any quadratic function can be represented using these transformations. The key is that you might have to complete the square. For example:
Say $f(x) = x^2 - 6x - 8$. Then we complete the square:
$f(x) = x^2 - 6x + 9 - 9 - 8 = (x^2 - 6x + 9) - 17 = (x-3)^2 - 17$
This tells us that this funciton is simply x^2 shifted to the right 3 units and downwards 17 units. Figure 5 confirms this. It also makes it simple to find the points where it intersects the x axis. We set $f(x) = 0$ and solve for x, which gives us $(x-3)^2 - 17 = 0$, so that $x = 3 \pm \sqrt{17}$. To find the y-intercept, we find $f(0) = 9 - 17 = -8$.

Figure 5



A slightly more complicated example:
$f(x) = 3x^2 + 5x - 4 = 3(x^2 + \frac{5}{3}x - \frac{4}{3}) = 3(x^2 + \frac{10}{6}x + \frac{25}{9} - \frac{25}{9} - \frac{4}{3}) = 3((x + \frac{5}{3})^2 - \frac{37}{9})$
So we can see that this function is that of x^2, shifted by $\frac{5}{3}$ to the left, then shifted by $\frac{37}{9}$ downwards, and finally scaled vertically by a factor of 3.

Practice exercises (not to be turned in)

For each of the following quadratics, use the methods explained above in order to analyze the function, sketch its graph, and identify its properties (try to be as thorough as possible in identifying their properties).

#P1. $f(x) = x^2 + 4x + 1$
#P2. $f(x) = -2x^2 - 8$
#P3. $f(x) = -x^2 - 6x + 12$
#P4. $f(x) = x^2 + x + 1$

Homework questions (to be turned in)

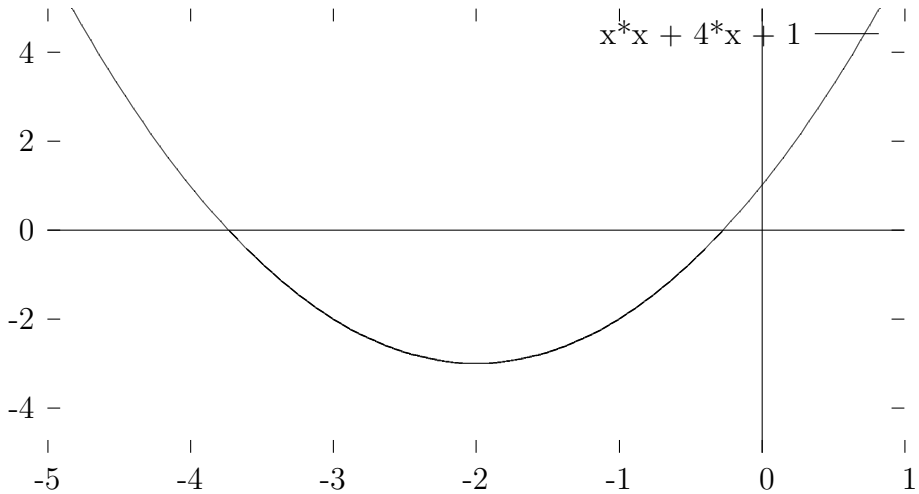
For each of the following quadratics, use the methods explained above in order to analyze the function, sketch its graph, and identify its properties.

#H1. $f(x) = x^2 - 10x$
#H2. $f(x) = -x^2 - 2x - 1$
#H3. $f(x) = \frac{1}{2}x^2 + 2x + 3$
#H4. $f(x) = 2x^2 + 12x + 3$

Solutions for practice exercises

#P1. $f(x) = x^2 + 4x + 1 = x^2 + 4x + 4 - 4 + 1 = (x+2)^2 - 3$. This means the function is x^2 shifted two units to the left and three down. This is show in figure 6:

Figure 6

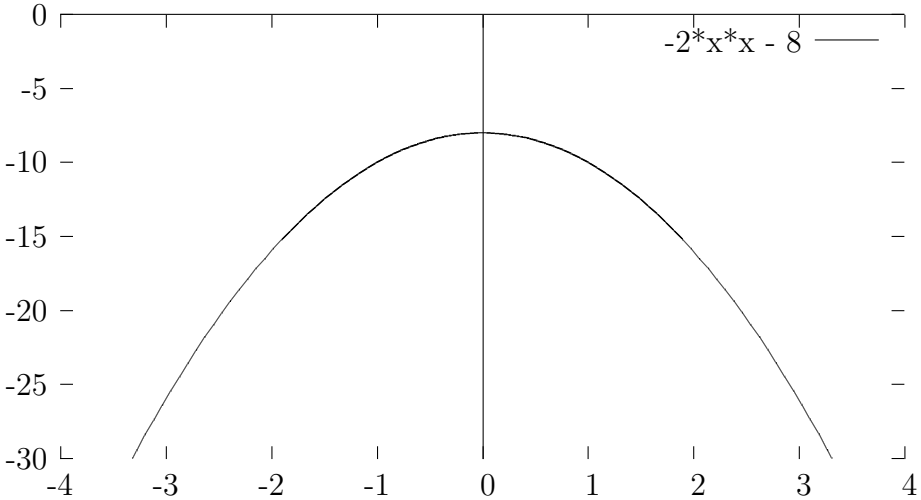


When sketching the function, we should also identify the noteworthy points. For example, the bottom of the curve is at $(-2, -3)$. It intercepts the y axis at $(0, 1)$ since $f(0) = 1$. To find where it intercepts the x axis, we set $f(x) = 0$ which gives us $(x+2)^2 - 3 = 0$. We solve this to get $x = -2 \pm \sqrt{3}$. So the two points are $(-2 - \sqrt{3}, 0)$ and $(-2 + \sqrt{3}, 0)$.
As far as the function's other properties, its domain and codomain are $\mathbb{R}$. It's not injective or surjective (and, therefore, it's not bijective either, though this can be left unsaid since it follows from those). It's symmetric around the line $x = -2$, decreasing for $x < -2$ and increasing for $x > 2$. Its range is $[-3, \infty[$. It's concave up, since it curves upwards as x increases. And its intersections with the axes are as described above.
For the next few exercises, I will just give the simplification and the graph.

#P2. $f(x) = -2x^2 - 8 = -2(x^2 + 4)$. Therefore, this is x^2 shifted up 4 units, then scaled vertically by a factor of two, and finally reflected along the x axis, as shown in figure 7. Notice that its maximum value is achieved at $(0, -8)$. This is because of the vertical scaling and because 8 is twice 4.

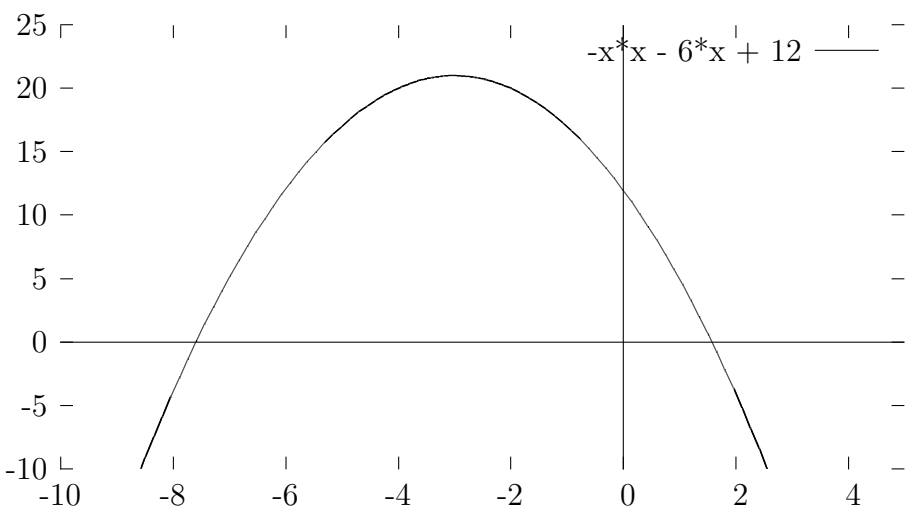
Also, note that another way to do it is to simply interpret $f(x)$ as it was originally: $f(x) = -2x^2 - 8$. You can interpret this as x^2 scaled by a factor of 2, then reflected along the x axis, and finally shifted down by 8 units. This illustrates the fact that different series of transformations can have the same overall result. Anything is fine so long as it helps you sketch the function and identify its properties.

Figure 7



#P3. $f(x) = -x^2 - 6x + 12 = -(x^2 + 6x) + 12 = -(x^2 + 6x + 9) + 9 + 12 = -(x + 3)^2 + 21$. We can interpret this as x^2 shifted three units to the left, then reflected along the x axis, and finally shifted 21 units up. This is shown in figure 8.

Figure 8



#P4. $f(x) = x^2 + x + 1 = x^2 + x + \frac{1}{4} - \frac{1}{4} + 1 = (x + \frac{1}{2})^2 + \frac{3}{4}$. This can be interpreted as x^2 shifted to the left by $\frac{1}{2}$ and then shifted up by $\frac{3}{4}$. Since this is a relatively simple transformation, I omit the graph.