Logarithmic functions: Part 1

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Logarithmic functions are inverse functions of exponential functions.
Recall what it means to be an inverse function:
A function, g, is an inverse function of f if $f(g(x)) = x$ and $g(f(x)) = x$ (for every value of x).

E.g. #1. $f(x) = x + 2$ and $g(x) = x - 2$ are inverses of each other. To see this, we can calculate
$f(g(x)) = f(x - 2) = (x - 2) + 2 = x$, and
$g(f(x)) = g(x + 2) = (x + 2) - 2 = x$.

E.g. #2. $f(x) = 4x - 5$ and $g(x) = \dfrac{x+5}{4}$ are inverse functions:
$f(g(x)) = 4g(x) - 5 = 4\dfrac{x+5}{4} - 5 = x + 5 - 5 = x$, and
$g(f(x)) = \dfrac{f(x)+5}{4} = \dfrac{(4x-5)+5}{4} = \dfrac{4x}{4} = x$.

If you don't remember well from 10A what inverse functions are and how to find them, you should watch the following videos at Khan Academy: Introduction to Function Inverses, Function Inverses Example 1, Function Inverses Example 2, and Function Inverses Example 3.
Not every function have an inverse function. For example, $f(x) = x^2$ doesn't have an inverse functions. You might think that $g(x) = \sqrt{x}$ is its inverse function, but that doesn't work, because, for example, $f(g(-2)) \neq -2$ (since you can't take a square root of a negative number). Only injective functions have inverse functions.

The **logarithmic function** is defined as the inverse of the exponential function. However, there are many different exponential functions, since the base can be any positive number that's not 1. So, for each exponential function, we define a corresponding logarithmic function.

Definition: If $a > 0$ and $a \neq 1$, then $\log_a x$, <u>the logarithmic function with base a</u>, is defined to be the inverse function of the exponential function a^x. This means:
$\log_a(a^x) = x$, and
$a^{\log_a x} = x$.
Put differently: $y = \log_a x$ if and only if $x = a^y$. If the base isn't indicated, it's assumed to be 10.

We can use this definition to isolate exponents. We do this by taking the logarithm of both sides of an equation.

E.g. #3. $3^x = 8$
$\log_3(3^x) = \log_3 8$
$x = \log_3 8$

E.g. #4. $2^{x^2+3} = 12$
$\log_2(2^{x^2+3}) = \log_2 12$
$x^2 + 3 = \log_2 12$
$x^2 = \log_3 12 - 3$ (notice the -3 is outside the log)
$x = \pm(\log_3 12 - 3)$

One can also go the other way around: applying an exponential function to an equation to isolate the argument of a logarithmic function.

E.g. $\log_6(x + 10) = 3$
$6^{\log_6(x+10)} = 6^3$
$x + 10 = 6^3$
$x = 6^3 - 10$

Range, domain of the logarithmic function

Since the logarithmic function is the inverse of the exponential function, its domain will be the exponential function's range, and its range will be the exponential function's domain.
Therefore:

$f(x) = \log_a x$ has domain $[0, \infty[$ and range $\mathbb{R}$.

As a result, it's very important that **you can't take a logarithm of zero or of a negative number**. This is, like dividing by zero, simply not allowed.

Practice exercises

Use the fact that the exponential function and the logarithmic function are inverses of each other to solve for x in the following equations:

#P1. $\left(\frac{1}{2}\right)^{-2x} = 9$

#P2. $\log_7(x+4) = 2$

#P3. $4^{\frac{x-6}{3}} = 5$

#P4. $0.2^{3x} = 8$

#P5. $3\log_2(4x) = 5$

Homework

Use the fact that the exponential function and the logarithmic function are inverses of each other to solve for x in the following equations:

#H1. $\log_6(3x) = 10$

#H2. $7^{2-5x} + 2 = 6$

#H3. $\log_{1.3}(4x-7) + 2 = 7$

#H4. $2^{x^2-1} = 5$

#H5. $0.7^{3x} + 10 = 23$

Solutions for practice problems

#P1. $\left(\frac{1}{2}\right)^{-2x} = 9$

There are different ways to do this one. One way is to change the base from $\frac{1}{2}$ to 2 using the power laws:

$\frac{1^{-2x}}{2^{-2x}} = 9$

$\frac{1}{2^{-2x}} = 9$

$2^{2x} = 9$

$\log_2(2^{2x}) = \log_2 9$

$2x = \log_2 9$

$x = \frac{\log_2 9}{2}$

The other way is use logarithm base $\frac{1}{2}$:

$\log_{\frac{1}{2}}\left(\left(\frac{1}{2}\right)^{-2x}\right) = \log_{\frac{1}{2}} 9$

$-2x = \log_{\frac{1}{2}}$

$x = \frac{-\log_{\frac{1}{2}} 9}{2}$

The two answers are actually equal, though we haven't yet learned the rules which allow us to see that.

#P2. $\log_7(x+4) = 2$

$7^{\log_7(x+4)} = 7^2$

$x + 4 = 49$

$x = 45$

#P3. $4^{\frac{x-6}{3}} = 5$

$\log_4 4^{\frac{x-6}{3}} = \log_4 5$

$\frac{x-6}{3} = \log_4 5$

$x - 6 = 3\log_4 5$

$x = 6 + 3\log 4_5$

#P4. $0.2^{3x} = 8$

$\log_{0.2} 0.2^{3x} = \log_{0.2} 8$

$3x = \log_{0.2} 8$

$x = \frac{\log_{0.2} 8}{3}$

#P5. $3\log_2(4x) = 5$

$\log_2(4x) = \frac{5}{3}$

$2^{\log_2(4x)} = 2^{\frac{5}{3}}$

$4x = 2^{\frac{5}{3}}$

$x = \frac{2^{\frac{5}{3}}}{4}$

$x = 2^{\frac{5}{3}-2}$

$x = 2^{\frac{-1}{3}}$