Trigonometric identities and trigonometric equations

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Trigonometric identities

An identity is like an equation, except that the two sides are equal for any values of the variables. This is in contrast to an equation, which is typically only satisfied by some values (which we find when we solve the equation).
Identities are often denoted using the symbol "$\equiv$" instead of the usual equality sign. However, this is not necessary.
Some trigonometric identities we have already seen:

1. Odd and even functions (obtained by reflecting the trigonometric circle around the x axis):
(a) $\cos(-\alpha) = \cos(\alpha)$ (cosine is even)
(b) $\sin(-\alpha) = -\sin(\alpha)$ (sine is odd)
(c) $\tan(-\alpha) = -\tan(\alpha)$ (tangent is odd)

2. Supplementary angles identities (obtained by reflecting the trigonometric circle around the y axis, and called 'supplementary angles identities' because supplementary angles are ones which add up to $180°$)
(a) $\cos(180° - \alpha) = -\cos(\alpha)$
(b) $\sin(180° - \alpha) = \sin(\alpha)$
(c) $\tan(180° - \alpha) = -\tan(\alpha)$

3. Cofunction identities (also known as complementary angle identities, since complementary angles are ones which add up to $90°$)

Sine and cosine are cofunctions:
(a) $\cos(90° - \alpha) = \sin\alpha$
(b) $\sin(90° - \alpha) = \cos\alpha$

Secant and cosecant are cofunctions:
(c) $\csc(90° - \alpha) = \sec\alpha$
(d) $\sec(90° - \alpha) = \csc\alpha$

Tangent and cotangent are cofunctions:
(e) $\cot(90° - \alpha) = \tan\alpha$
(f) $\tan(90° - \alpha) = \cot\alpha$

4. Tangent and cotangent in terms of sine and cosine
(a) $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha}$
(b) $\cot\alpha = \dfrac{\cos\alpha}{\sin\alpha}$
To these, we add:

5. The Pythagorean identities:
(a) $\sin^2\alpha + \cos^2\alpha = 1$
(b) $\tan^2\alpha + 1 = \sec^2\alpha$
(c) $1 + \cot^2\alpha = \csc^2\alpha$

The Pythagorean identities

(a) Remember the definitions from SOHCAHTOA, we can prove the first Pythagorean identity:

Suppose we have a right-angles triangle, with an angle α. Then:

$\sin^2\alpha + \cos^2\alpha$

$= \left(\frac{\text{opp}}{\text{hyp}}\right)^2 + \left(\frac{\text{adj}}{\text{hyp}}\right)^2$

$= \frac{\text{opp}^2}{\text{hyp}^2} + \frac{\text{adj}^2}{\text{hyp}^2}$

$= \frac{\text{opp}^2 + \text{adj}^2}{\text{hyp}^2}$

$= \frac{\text{hyp}^2}{\text{hyp}^2}$ (using the Pythagorean theorem)

This is for angles between $0°$ and $90°$. But it can be seen to also work for other angles, since we have found, using the trigonometric circle, that $\sin\alpha$ will be either $\sin\beta$ or $-\sin\beta$, where β is the reference angle of α, and the same applies for cosine. The minus sines disappear when we square the sine and the cosine, so it works out the same in the end.

From this basic Pythagorean identity, we can get the other two:

(b) Dividing both sides by $\cos^2\alpha$:

$\frac{\sin^2\alpha}{\cos^2\alpha} + \frac{\cos^2\alpha}{\cos^2\alpha} = \frac{1}{\cos^2\alpha}$

$\tan^2\alpha + 1 = \sec^2\alpha$

(c) Dividing both sides by $\sin^2\alpha$:

$\frac{\sin^2\alpha}{\sin^2\alpha} + \frac{\cos^2\alpha}{\sin^2\alpha} = \frac{1}{\sin^2\alpha}$

$1 + \cot^2\alpha = \csc^2\alpha$

We will use some of these identities later on in solving trigonometric equations or simplifying expressions. For now we just need to be acquainted with them.

Trigonometric equations

A trigonometric equation is just an equation involving trigonometric functions. In fact, we have already seen some.

E.g.

(a) $\cos\alpha = \frac{1}{2}$

(b) $\sec\alpha = 1$

etc.

These came up in the context of finding preimages, and we already know how to solve them.

Next to these, the simplest are ones which can be reduced to such equations.
E.g. #1. $6\sin\alpha - 2 = 0$
So $6\sin\alpha = 2$
$\sin\alpha = \frac{1}{3}$, which we know how to solve.
So we'll move on to more complicated ones.

Quadratic trigonometric equations

This involve a product of two trigonometric functions (or the same trigonometric function multiplied by itself, e.g. $\sin^2 x$).
The general strategy will be to move everything to one side of the equation, and factorize it so that it's a product of two terms. We can then say that one of them has to be zero, and do each one separately. In a way, this is analogous to how we solve quadratic equations using factorization. Just as with quadratic equations, it's sometimes possible to solve it more easily without factorizing (e.g. $x^2 = 4$ can be solved directly to give $x = \pm 2$).

E.g. #1. Find all the solutions of $\sin\alpha\cos\alpha - 2\cos\alpha = 0$ in the interval $[0, 360°]$.
We factorize this, and we get:
$\cos\alpha(\sin\alpha - 2) = 0$
So this tells us $\cos\alpha = 0$ or $\sin\alpha - 2 = 0$.
This means $\cos\alpha = 0$ or $\sin\alpha = 2$. So the solutions will be the preimages of 0 under cosine, and the preimages of 2 under sine.
Now, in the interval $[0, 360°]$, $\cos\alpha = 0$ for $\alpha = 90°$ and $\alpha = 270°$.
On the other hand, $\sin\alpha = 2$ has no solutions, because sine goes between -1 and 1.
So our solutions for the original equation will be 90° and 270°.

E.g. #2. Find all the solutions of $\sin^2\alpha = \frac{1}{4}$ in $[-\pi, \pi]$.
Here, we can just take the square root of both sides, but we must remember the plus or minus:
$\sin\alpha = \pm\frac{1}{2}$
So our solutions will be the preimages (under sine) of $\frac{1}{2}$ and of $\frac{1}{2}$.
The preimages of $\frac{1}{2}$ in $[-\pi, \pi]$ are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.
The preimages of $-\frac{1}{2}$ in $[-\pi, \pi]$ are $-\frac{\pi}{6}$ and $-\frac{5\pi}{6}$.
So our solutions will be $-\frac{5\pi}{6}$, $-\frac{\pi}{6}$, $\frac{\pi}{6}$, and $\frac{5\pi}{6}$.

E.g. #3. Find all the solutions of $\cos^2\alpha + 2 = -3\cos\alpha$ in $[0, 4\pi]$.
First, we move everything to the left, and then factorize:
$\cos^2\alpha + 3\cos\alpha + 2 = 0$
$(\cos\alpha + 2)(\cos\alpha + 1) = 0$
Therefore, either $\cos\alpha + 2 = 0$ or $\cos\alpha + 1 = 0$
To have $\cos\alpha + 2 = 0$ would mean $\cos\alpha = -2$. But this has no solutions, because cosine goes between -1 and 1.
To have $\cos\alpha + 1 = 0$ would mean $\cos\alpha = -1$. So we need to find the preimages, under cosine, of -1. The basic solution is $\frac{3\pi}{2}$, and it repeats every 2π. So in the interval $[0, 4\pi]$, that means our solutions will be $\frac{3\pi}{2}$ and $\frac{7\pi}{2}$.