Trigonometric equations requiring the use of trigonometric identities

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In solving trigonometric equations, we sometimes need to first use a trigonometric identity. There are two main cases:

A. Using $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$

This will typically occur when we have an equation with two terms, one $\sin\alpha$ and the other $\cos\alpha$ (they can have coefficients). In such cases, there is simply no way to factorize, so we must do something else.

E.g. #1. Solve $\sin\alpha = 2\cos\alpha$, with $\alpha \in [0, 2\pi]$.
What we want to do is get $\frac{\sin\alpha}{\cos\alpha}$ so that we can replace it with $\tan\alpha$ and make things simpler. To do this, we divide both sides by $\cos\alpha$. However, this is only allowed if $\cos\alpha \neq 0$. So we need to consider separately the case of $\cos\alpha = 0$.
Case 1: $\cos\alpha \neq 0$.
Then we divide both sides by $\cos\alpha$:
$\frac{\sin\alpha}{\cos\alpha} = 2$
$\tan\alpha = 2$
One solution will be $\alpha = \tan^{-1} 2 = 63°$ (rounded to the nearest degree). Since tangent has a period of 180°, we can get another solution by adding 180° to 63°, which gives 243°. Adding 180° again or subtracting it from 63° takes us outside $[0, 2\pi]$, so these are the only solutions.

Case 2: $\cos\alpha = 0$.
In this case, we must have $\alpha = 90°$ or $\alpha = 270°$. However, neither of these satisfy the original equation, so they're not solutions.

Therefore, the solutions are $\alpha = 63°$ and $\alpha = 243°$.

B. Using $\sin^2\alpha + \cos^2\alpha = 1$

The Pythagorean identity will be useful when we have $\sin^2\alpha$ or $\cos^2\alpha$ in our equation. However, we don't always have to use it, since in some cases we can just factorize (e.g. $\sin^2\alpha + 3\sin\alpha + 2 = (\sin\alpha + 2)(\sin\alpha + 1)$). We will have to use it in cases where we can't factorize, or when it will clearly make things simpler. This will typically be when we have $\cos^2\alpha$ and $\sin\alpha$, when we have $\sin^2\alpha$ and $\cos\alpha$, or when we have both $\cos^2\alpha$ and $\sin^2\alpha$.

E.g. #2. Solve $\sin^2\alpha - 2\cos^2\alpha = 1$, with $\alpha \in [-\pi, \pi]$.
Here if we move everything to the left we get something we can't factorize. So we can use the Pythagorean identity to eliminate either the $\sin^2\alpha$ or the $\cos^2\alpha$. Either one works. Since $\sin^2\alpha + \cos^2\alpha = 1$, we know $\cos^2\alpha = 1 - \sin^2\alpha$. We can substitute this into our equation:
$\sin^2\alpha - 2(1 - \sin^2\alpha) = 1$
$\sin^2\alpha - 2 + 2\sin^2\alpha) = 1$
$3\sin^2\alpha = 3$
$\sin^2\alpha = 1$
$\sin\alpha = \pm 1$, so we have two cases:
Case 1: $\sin\alpha = 1$. This means $\alpha = 90°$.

Case 2: $\sin\alpha = -1$. This means $\alpha = -90°$.
Therefore, our solutions are $90°$ and $-90°$.

E.g. #3. Solve $2\cos^2\alpha - \sin\alpha - 1 = 0$, with $\alpha \in [0, 2\pi]$.
We have no way to factorize this. So we use $\sin^2\alpha + \cos^2\alpha = 1$ to eliminate the term with $\cos^2\alpha$. The identity can be rewritten $\cos^2\alpha = 1 - \sin^2\alpha$, which we can then substitute into our equation:
$2\cos^2\alpha - \sin\alpha - 1 = 0$, which gives:
$2(1 - \sin^2\alpha) - \sin\alpha - 1 = 0$
$2 - 2\sin^2\alpha - \sin\alpha - 1 = 0$
$-2\sin^2\alpha - \sin\alpha + 1 = 0$
To make it easier to factorize, I'll multiply both sides by -1.
$2\sin^2\alpha + \sin\alpha - 1 = 0$
$(2\sin\alpha - 1)(\sin\alpha + 1) = 0$
Therefore, $2\sin\alpha - 1 = 0$ or $\sin\alpha + 1 = 0$.
Case 1: $2\sin\alpha - 1 = 0$. Therefore, $\sin\alpha = \frac{1}{2}$. This has $\alpha = 30°$ as one solution. The other one, which can be found using the trigonometric circle or the graph, is $150°$.
Case 2: $\sin\alpha + 1 = 0$. Therefore, $\sin\alpha = -1$, which gives $\alpha = 270°$.
Therefore, our solutions are $30°$, $150°$, and $270°$.

<u>Homework for Monday</u>

#1. Solve $\cos\alpha + 4\sin\alpha = 0$, with $\alpha \in [-\pi, \pi]$ (round your answers to the nearest degree).

#2. Solve $\cos^2\alpha - \sin\alpha + 1 = 0$, with $\alpha \in [0, 2\pi]$.

#3. Solve $3\sin^2\alpha = 1 + 2\cos^2\alpha$, with $\alpha \in [-\pi, \pi]$ (round your answers to the nearest degree).