Logarithmic functions: non-maximal domains and ranges

Mr. Neeman, 10B, Sept. 14, 2011.

Given a logarithmic mapping (e.g. $f(x) = \log_3 x$), we can find the range if we're given the domain, or find the range if we're given the domain. For this, we need to know whether the function is increasing or decreasing, which we can tell from the graph, or from calculating some images or preimages (or from starting with a basic function and taking into account the transformations).

E.g. #1. Consider $f(x) = \log_3 x$.
(a) Find the function's range if the domain is $[9, 81[$.
We need to find the images of the end-points:
$f(9) = \log_3 9 = \log_3(3^2) = 2$
$f(81) = \log_3 81 = \log_3(3^4) = 4$
Now, the range will have these images as its endpoints. Since 4 is greater than 2 (so the function is increasing), the 2 will come first in the interval notation. Since the domain includes 9, but doesn't include 81, the range will include 2, but not 4. So the range is $[2, 4[$

(b) Find the function's range if the domain is $[5, \infty[$.
$f(5) = \log_3 5$
We can't calculate this as it is, but we know it will be somewhere between 1 and 2, since $\log_3 3 = 1$ and $\log_3 = 2$. We also know that $\log_3 x$ is increasing and unbounded (as x goes to infinity, so does y. So the range will be $[\log_3 5, \infty[$

(c) Find the function's domain if the range is $]-\infty, 3]$.
For this part (and the others as well), it's helpful to have in front of you the graph of $\log_3 x$ on its maximal domain ($[0, \infty[$). It has a vertical asymptote at $x = 0$, in which the function goes to $-\infty$. So the domain will be starting at 0 (without including it), since that's where it has values which are very negative. It will then go on until it gets to $f(x) = 3$, which means $\log_3 x = 3$. This gives us $x = 3^3 = 27$. So the domain is $]0, 27]$.

E.g. #2. Consider the mapping $f(x) = 1 + \log_{\frac{1}{2}} x$
(a) Find the range if the domain is $]0, 1[$.
The function has a vertical asymptote, going upwards, at $x = 0$. So as x gets closer and closer to 0, $f(x)$ goes to ∞.
$f(1) = 1 + \log_{\frac{1}{2}} 1 = 1 + 0 = 1$
So the endpoints will be ∞ and 1. The lower one always goes first. So the range will be $[1, \infty[$.

(b) Find the domain if the range is $[-1, 1[$.

To find the preimage of -1: $1 + \log_{\frac{1}{2}} x = -1$. therefore $\log_{\frac{1}{2}} x = -2$, and $x = \left(\frac{1}{2}\right)^{-2} = 4$.

To find the preimage of 1, we can use our result from part (a): $f(1) = 1$. So our domain will have endpoints 1 and 4. Now, 1 corresponds to 1, so it will be excluded, whereas 4 corresponds to -1, so it will be included. So the domain is $]1, 4]$.

Homework for Friday

#1. Consider the mapping $f(x) = \log_5 x - 2$.
(a) Sketch the function's graph if the domain is $]0, \infty[$.
(b) Is the function increasing or decreasing? (this doesn't depend on the domain)
(c) What is the function's concavity? (this doesn't depend on the domain)
(d) Find the function's range if the domain is $[\frac{1}{5}, 5[$.
(e) Find the function's range if the domain is $]0, 1]$.
(f) Find the function's domain if the range is $]2, \infty[$.
(g) Find the function's domain if the range is $]-1, 2]$.

#2. Consider the mapping $f(x) = \log_2 x$
(a) Sketch the function's graph if the domain is $]0, \infty[$.
(b) Is the function increasing or decreasing? (this doesn't depend on the domain)
(c) What is the function's concavity? (this doesn't depend on the domain)
(d) Find the function's range if the domain is $[7, \infty[$.
(e) Find the function's range if the domain is $[8, 16[$.
(f) Find the function's domain if the range is $]-3, 2]$.
(g) Find the function's domain if the range is $]-\infty, 0[$.