Practice on inequalities

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Solve the following inequalities.

Simple fractional inequalities (This the level of fractional iequalities you should expect on the quiz and exams. But keep in mind that they're simple because of the way things work out, so that a mistake in the early part of the problem could give you something very difficult, so you must work very carefully.)

#1. $2x + 2 \geq \dfrac{2x^2 + 2x + 6}{x + 1}$

#2. $\dfrac{x(x^2 - 6x + 20)}{x - 2} < 0$

#3. $\dfrac{x^2 - 3}{x - 2} \leq 2$

#4. $\dfrac{2}{x} \geq -x$

Quadratic inequalities

#5. $x + 3 < 2x^2$

#6. $2x^2 + 12x + 18 \leq 0$

#7. $x^2 + 5 \geq 4x$

#8. $x^2 + 7x < -10$

#9. $x^2 + 4x + 2 \leq 0$

Polynomial inequalities

#10. $x^3 + 4 \leq 3x^2$

#11. $x^3 - x^2 + 2x - 2 < 0$

#12. $x^3 + 9x > 6x^2$

More challenging fractional inequalities (these are the most challenging, harder than what I'd put on a quiz or exam, but good practice to get to the point where you're on top of the material so that the quiz and the exams will then be easy for you. I recommend solving them, after all the other types)

#13. $x \leq \dfrac{2x}{x - 1}$

#14. $x - 4 \leq \dfrac{-5}{x + 2}$

#15. $\dfrac{-5}{x - 1} \geq \dfrac{x^2 + 1}{x^2 - 1}$

(this one is rather complicated, nothing nearly this hard would be on the quiz or midterm)

#16. $x + \dfrac{4}{x^2 + 5x} > \dfrac{1}{x + 5}$

Solutions

Simple fractional inequalities

#1. $2x+2 \geq \dfrac{2x^2+2x+6}{x+1}$

$\dfrac{(2x+2)(x+1)}{x+1} \geq \dfrac{2x^2+2x+6}{x+1}$

$\dfrac{2x^2+4x+2}{x+1} \geq \dfrac{2x^2+2x+6}{x+1}$

$\dfrac{2x-4}{x+1} \geq 0$

$\dfrac{x-2}{x+1} \geq 0$

So the key values are $x=-1$ and $x=2$. The last row is the product of the second and third rows.

	$x<-1$	$x=-1$	$-1<x<2$	$x=2$	$x>2$
$x+1$	-	0	+	+	+
$\dfrac{1}{x+1}$	-	undef.	+	+	+
$x-2$	-	-	-	0	+
$\dfrac{x+1}{x-2}$	+	undef.	-	0	+

We're looking for it to be positive or zero, so $x<-1$ or $x \geq 2$.

#2. $\dfrac{x(x^2-6x+20)}{x-2} < 0$

First, we look at the quadratic term to see whether it can be factorized. Completing the square, we get $x^2-6x+20=(x-3)^2+11$, which means it's irreducible (it has no roots). So it can never be zero, and we can divide both sides by it, getting:

$\dfrac{x}{x-2} < 0$

The key values are $x=0$ and $x=2$. The last row is the product of the first and third.

	$x<0$	$x=0$	$0<x<2$	$x=2$	$x>2$
x	-	0	+	+	+
$x-2$	-	-	-	0	+
$\dfrac{1}{x-2}$	-	-	-	0	+
$\dfrac{x}{x-2}$	+	0	-	undef.	+

We're looking for it to be negative, which means $0<x<2$.

#3. $\dfrac{x^2-3}{x-2} \leq 2$

$\dfrac{x^2-3}{x-2} - 2 \leq 0$

$\dfrac{x^2-3-2x+4}{x-2} \leq 0$

$\dfrac{x^2-2x+1}{x-2} \leq 0$

$\dfrac{(x-1)^2}{x-2} \leq 0$

So the key values are $x = 1$ and $x = 2$. The last row is the product of the second and fourth.

	$x < 1$	$x = 1$	$1 < x < 2$	$x = 2$	$x > 2$
$x - 1$	-	0	$+$	$+$	$+$
$(x-1)^2$	$+$	0	$+$	$+$	$+$
$x - 2$	-	-	-	0	$+$
$\dfrac{1}{x-2}$	-	-	-	undef.	$+$
$\dfrac{(x-1)^2}{x-2}$	-	0	-	undef.	$+$

We're looking for it to be 0 or negative, so we get $x < 2$.

#4. $\dfrac{2}{x} \geq -x$

$\dfrac{2}{x} + x \geq 0$

$\dfrac{2 + x^2}{x} \geq 0$

$2 + x^2$ is an irreducible polynomial, so we can divide by both sides, getting

$\dfrac{1}{x} \geq 0$

But we know that $\dfrac{1}{x}$ always has the same sign as x (if $x \neq 0$). So this tells us the solution is $x > 0$. We can also do it with a small table:

	$x < 0$	$x = 0$	$x > 0$
x	-	0	$+$
$\dfrac{1}{x}$	-	undef.	$+$

We're looking for it to be 0 or positive, which gives us $x > 0$.

Quadratic inequalities

#5. $x + 3 < 2x^2$

$0 < 2x^2 - x - 3$

$0 < (2x - 3)(x + 1)$

The key values are $x = -1$ and $x = \dfrac{3}{2}$. Doing the table, we find the answer is $x < -1$ or $x > \dfrac{3}{2}$.

#6. $2x^2 + 12x + 18 \leq 0$

We can divide by 2 (since 2 is positive, the sign stays the same)

$x^2 + 6x + 9 \leq 0$

$(x + 3)^2 \leq 0$

Now, a square is zero at its root and positive elsewhere. So the inequality will be satisfied just when $x + 3 = 0$. So the answer is $x = -3$.

#7. $x^2 + 5 \geq 4x$
$x^2 - 4x + 5 \geq 0$
$(x-2)^2 + 1 \geq 0$
Now, the square can never be less than zero, and 1 is positive, so the left hand side will always be positive. Therefore, the inequality is satisfied for all real numbers.

#8. $x^2 + 7x < -10$
$x^2 + 7x + 10 < 0$
$(x+5)(x+2) < 0$
The key values here are $x = -5$ and $x = -2$. Doing the table, we find the answer is $-5 < x < -2$.

#9. $x^2 + 4x + 2 \leq 0$
$(x+2)^2 - 2 \leq 0$
$(x + 2 + \sqrt{2})(x + 2 - \sqrt{2}) \leq 0$
So the key values are $x = -2 - \sqrt{2}$ and $x = -2 + \sqrt{2}$. Doing the table we find the answer is $-2 - \sqrt{2} \leq x \leq -2 + \sqrt{2}$.

Polynomial inequalities

#10. $x^3 + 4 \leq 3x^2$
$x^3 - 3x^2 + 4 \leq 0$
If there are rational roots, they must be divisors of 4: 1, -1, 2, -2, 4, or -4. We can try 1, but it isn't a root. We can then try -1, and we see that it is a root. Doing the synthetic or polynomial division by $x + 1$, we then get:
$(x+1)(x^2 - 4x + 4) \leq 0$
We can factorize the quadratic:
$(x+1)(x-2)^2 \leq 0$
So our key values are $x = -1$ and $x = 2$. We can then do the table. The last row is the product of the first and the third.

	$x < -1$	$x = -1$	$-1 < x < 2$	$x = 2$	$x > 2$
$x + 1$	-	0	+	+	+
$x - 2$	-	-	-	0	+
$(x-2)^2$	+	+	+	0	+
$(x+1)(x-2)^2$	-	0	+	0	+

We're looking for it to be 0 or negative, which tells us the answer is $x \leq -1$ or $x = 2$.

#11. $x^3 - x^2 + 2x - 2 < 0$
This can factorized, either by grouping or by trying the divisors of 2 (1, -1, 2, and -2) and then doing polynomial division. Grouping is easier if you can see it:
$x^2(x-1) + 2(x-1) < 0$
$(x^2 + 2)(x-1) < 0$
Now, $x^2 + 2$ is an irreducible polynomial, so we can divide both sides of the inequality by it, getting:
$x - 1 < 0$
Therefore, $x < 1$.

#12. $x^3 + 9x > 6x^2$
$x^3 - 6x^2 + 9x > 0$
$x(x^2 - 6x + 9) > 0$
$x(x-3)^2 > 0$
So the key values are $x = 0$ and $x = 3$. We can do the table. The last row is the product of the first and third rows.

	$x < 0$	$x = 0$	$0 < x < 3$	$x = 3$	$x > 3$
x	-	0	+	+	+
$x - 3$	-	-	-	0	+
$(x-3)^2$	+	+	+	0	+
$x(x-3)^2$	-	0	+	0	+

We're looking for it to be positive, which gives $0 < x < 3$ or $x > 3$.

<u>More challenging fractional inequalities</u>

#13. $x \le \dfrac{2x}{x-1}$
$0 \le \dfrac{2x}{x-1} - x$
$0 \le \dfrac{2x - x(x-1)}{x-1}$
$0 \le \dfrac{3x - x^2}{x-1}$
$\dfrac{x^2 - 3x}{x-1} \le 0$
$\dfrac{x(x-3)}{x-1} \le 0$
So our key values are $x = 0$, $x = 3$, and $x = 1$. We can do the table, keeping in mind that the last row is the product of the first, second, and fourth rows.

	$x < 0$	$x = 0$	$0 < x < 1$	$x = 1$	$1 < x < 3$	$x = 3$	$x > 3$
x	-	0	+	+	+	+	+
$x - 3$	-	-	-	-	-	0	+
$x - 1$	-	-	-	0	+	+	+
$\dfrac{1}{x-1}$	-	-	-	undef.	+	+	+
$\dfrac{x(x-3)}{x-1}$	-	0	+	undef.	-	0	+

The inequality is satisfied where it's 0 or negative. So this means $x \le 0$ or $1 < x \le 3$.

#14. $x - 4 \leq \dfrac{-5}{x+2}$

$x - 4 + \dfrac{5}{x+2} \leq 0$

$\dfrac{(x-4)(x+2)+5}{x+2} \leq 0$

$\dfrac{x^2 - 2x - 3}{x+2} \leq 0$

$\dfrac{(x-3)(x+1)}{x+2} \leq 0$

So our key values are $x = -2$, $x = -1$, and $x = 3$. We can do the table, keeping in mind that the last row is the product of the second, third, and fourth rows.

	$x < -2$	$x = -2$	$-2 < x < -1$	$x = -1$	$-1 < x < 3$	$x = 3$	$x > 3$
$x + 2$	-	0	+	+	+	+	+
$\dfrac{1}{x+2}$	-	undef.	+	+	+	+	+
$x + 1$	-	-	-	0	+	+	+
$x - 3$	-	-	-	-	-	0	+
$\dfrac{(x-3)(x+1)}{x-2}$	-	undef.	+	0	-	0	+

The inequality is satisfied where it's 0 or negative. So this means $x < -2$ or $-1 \leq x \leq 3$.

#15. $\dfrac{-5}{x-1} \geq \dfrac{x^2+1}{x^2-1}$

$0 \geq \dfrac{x^2+1}{x^2-1} + \dfrac{5}{x-1}$

$0 \geq \dfrac{x^2+1+5(x+1)}{(x+1)(x-1)}$

$0 \geq \dfrac{x^2+5x+6}{(x+1)(x-1)}$

$0 \geq \dfrac{(x+2)(x+3)}{(x+1)(x-1)}$

So our key values are $x = -3$, $x = -3$, $x = -1$, and $x = 1$. This means we get a pretty big table! The last row is the product of the first, second, fourth, and sixth rows.

	$x < -3$	$x = -3$	$-3 < x < -2$	$x = -2$	$-2 < x - 1$	$x = -1$	$-1 < x < 1$	$x = 1$	$x > 1$
$x + 3$	-	0	+	+	+	+	+	+	+
$x + 2$	-	-	-	0	+	+	+	+	+
$x + 1$	-	-	-	-	-	0	+	+	+
$\dfrac{1}{x+1}$	-	-	-	-	-	undef.	+	+	+
$x - 1$	-	-	-	-	-	-	-	0	+
$\dfrac{1}{x-1}$	-	-	-	-	-	-	-	undef.	+
$\dfrac{(x+3)(x+2)}{(x+1)(x-1)}$	+	0	-	0	+	undef.	-	undef	+

We're looking for it to be 0 or negative, which means $-3 \leq x \leq -2$ or $-1 < x < 1$.

#16. $x + \frac{4}{x^2+5x} > \frac{1}{x+5}$
$\frac{x(x+5)+4-x}{x(x+5)} > 0$
$\frac{x^2+4x+4}{x(x+5)} > 0$
$\frac{(x+2)^2}{x(x+5)} > 0$
So our key values are $x=-5$, $x=-2$, and $x=0$. The last row is the product of the second, fourth, and sixth rows.

	$x<-5$	$x=-5$	$-5<x<-2$	$x=-2$	$-2<x<0$	$x=0$	$x>0$
$x+5$	-	0	+	+	+	+	+
$\frac{1}{x+5}$	-	undef.	+	+	+	+	+
$x+2$	-	-	-	0	+	+	+
$(x+2)^2$	+	+	+	0	+	+	+
x	-	-	-	-	-	0	+
$\frac{1}{x}$	-	-	-	-	-	undef.	+
$\frac{(x+2)^2}{x(x+5)}$	+	undef.	-	0	-	undef.	+

We're looking for it to be positive, so $x<-5$ or $x>0$.