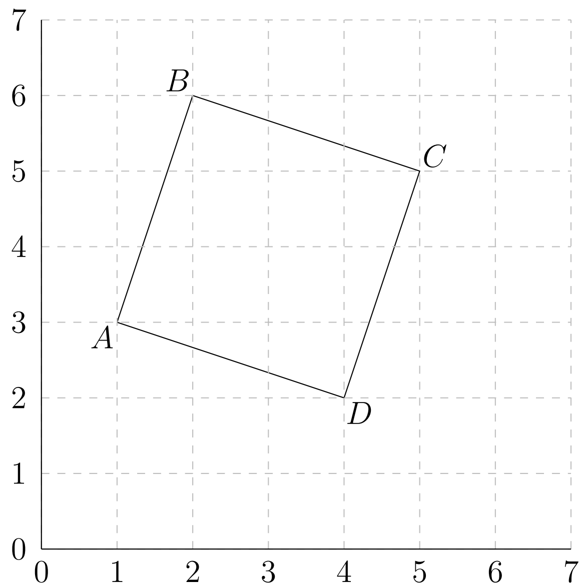
Analytic geometry: triangles and rectangles
Mr. Neeman, 10A. November 11, 2011

We can put to use our work with lines, distances, and midpoints, doing various calculations with geometrical figures. We will need the usual formulas for their areas:

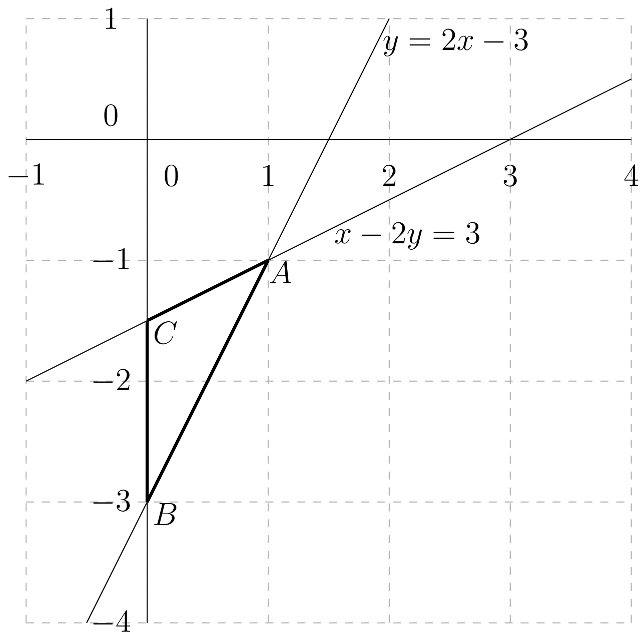
Area of a rectangle = length $\times$ width.
Area of a triangle = base $\times$ height.
Also, Heron's formula for the area of a triangle can be useful:
Area of a triangle $= \sqrt{s(s-a)(s-b)(s-c)}$, where a, b, c are the sides and $s = \frac{a+b+c}{2}$ is the semiperimeter.

E.g. #1. Find the area of the rectangle with vertices $A = (1,3), B = (2,6), C = (5,5)$, and $D = (4,2)$.
We can, first, draw a diagram to get an idea of the picture.



To find the area, we need the lengths of the sides (we just need any two adjacent ones, since opposite sides are equal to each other).
$AB = \sqrt{(6-3)^2 + (2-1)^2} = \sqrt{9+1} = \sqrt{10}$
$BC = \sqrt{(5-2)^2 + (5-6)^2} = \sqrt{9+1} = \sqrt{10}$
We see that this rectangle is actually a square. It's area is $\sqrt{10}\sqrt{10} = 10$.

E.g. #2. Consider the triangle formed by the lines $x - 2y = 3$, $y = 2x - 3$, and the y axis. Find its perimeter and its area.
We need to find the lengths of the sides in order to find the perimeter. For that, we need to find the vertices, which are the points of intersection of pairs of sides. This will also enable us to draw the diagram. We'll do each one in turn:
To find the intersection of $x - 2y = 3$ and $y = 2x - 3$, we can substitute the second into the first:
$x - 2(2x - 3) = 3$
$x - 4x + 6 = 3$
$-3x = -3$
$x = 1$
Substituting this back into our second line, we get $y = 2(1) - 3 = -1$. So one vertex, call it A, will be $(1, -1)$.
The intersection of $y = 2x - 3$ with the y axis is just -3, since that's what we have for y when $x = 0$. So the second vertex, B, will be $(0, -3)$.
To find the intersection of $x - 2y = -3$ with the y axis, we set $x = 0$, which gives us $-2y = 3$, so that $y = -\frac{3}{2}$. So the third vertex, C, will be $(0, -\frac{3}{2})$.
We can see the triangle formed by these lines represented in the diagram. The triangle itself is shown in bold.



Now, to find the perimeter, we find the length of each side:
$AB = \sqrt{(1-0)^2 + (-1-(-3))^2} = \sqrt{5}$
$BC = -\frac{3}{2} - (-3) = \frac{3}{2}$ (since this side is part of a vertical line).
$AC = \sqrt{(1-0)^2 + (-1-(-\frac{3}{2}))^2} = \sqrt{1 + \frac{1}{4}} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$
Therefore the perimeter is: $\sqrt{5} + \frac{3}{2} + \frac{\sqrt{5}}{2} = \frac{3+3\sqrt{5}}{2}$.
To find the area, we can multiply the base by the height, because one of our sides is the y axis, so it's easy to find. The base will be the side AB, which is $\frac{3}{2}$. The height will be the horizontal distance from the y axis to the vertex A, which is 1.
Area $= \frac{1}{2}\left(\frac{3}{2}\right)(1) = \frac{3}{4}$.
We could also find the area using Heron's formula, though in this case it'd be considerably more complicated that what we just did.

Exercises

#1. Consider the rectangle with vertices $A = (-2, 2), B = (1, 4), C = (-1, 7)$, and $D = (-4, 5)$.
(a) Draw a labelled diagram representing the rectangle.
(b) Find the equations of each of the sides.
(c) Verify that it is a rectangle by checking that opposite sides are parallel to each other and the pairs of adjacent sides are perpendicular to each other.
(d) Find its area.
(e) Find the length of its diagonal.
(f) Find the rectangle's center.

#2. Suppose you have a triangle with vertices $A = (1, 1), B = (2, 3)$, and $C = (4, -6)$, and that we take the side AB to be our base.
(a) Draw a labelled diagram representing this triangle.
(b) Find the equation of the line containing the base.
(c) Find the equation of the line containing the height.
(d) Find the point at which the base intersects the height.
(e) Find the height.
(f) Find the triangle's area.

#3. Find the equation of a line which is perpendicular to $x = 3$ and which passes through the point $(-2, 5)$.

#4. Suppose the line l_1 passes through the points $(0, 4)$ and $(-2, -1)$. Find the equation of the line which is parallel to l_1 and which passes through the point $(3, 1)$.

#5. Find the area of the triangle formed by the line $x + 3y = -9$ and the two axes.