Finding intersections of lines
Mr. Neeman, 10A. November 4, 2011

We have already seen how to find the intersections of a line (or any function's graph) with the axes:

To find the intersection of $y = f(x)$ with the y axis, we set $x = 0$, and then solve for y.
To find the intersection of $y = f(x)$ with the x axis, we set $y = 0$, and then solve for x.

Now that we know what the equation of a horizontal line or a vertical line is, we can see these procedures from a broader perspective. Remembering that the equation of the x axis is $y = 0$ and the equation of the y axis is $x = 0$:

The intersection of the graph of $y = f(x)$ with the y axis satisfies both $y = f(x)$ (it lies on the graph of f) and $x = 0$ (it lies on the y axis).
The intersection of the graph of $y = f(x)$ with the x axis satisfies both $y = f(x)$ (it lies on the graph of f) and $y = 0$ (it lies on the y axis).

We can generalize:

> If we have two lines or curves, each given by a certain equation, their point or points of intersection will satisfy both of their equations.

Limiting ourselves now to straight lines, we can say:

> Given two straight lines, each given by an equation, their point or points of intersection will be all those points which satisfy both equations. To find them, we solve the equations together (as a system, or simultaneously).

E.g. #1. To find the intersection of $y = 2x + 1$ and $y = -x - 5$, we need to solve the system consisting of both equations:

$$\begin{cases} y = 2x + 1 \\ y = -x - 5 \end{cases}$$

This can be solved very easily by substitution. The second equation tells us $y = -x - 5$, so we can substitute that into the first equation:

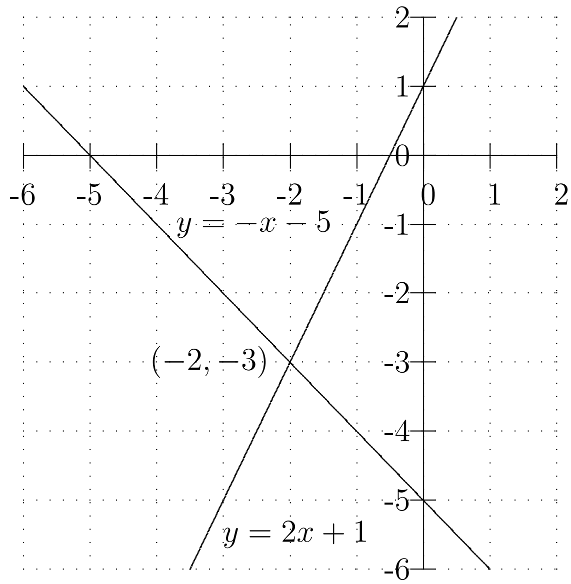
$-x - 5 = 2x + 1$
$-6 = 3x$
$-2 = x$

Now that we have the value of x, we can find y by substituting x into either of the equations. Using the second one:

$y = -(-2) - 5 = -3$.

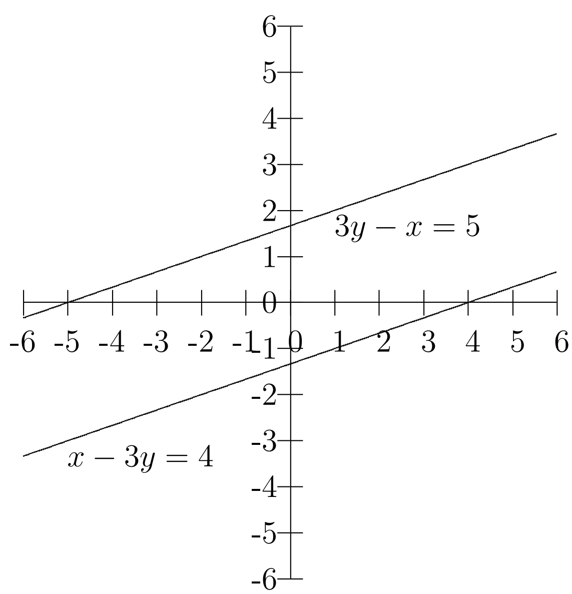
So the point of intersection is $(-2, -3)$. We can check that, in fact, this point satisfies each of the two equations by substituting these back in. The lines are shown in the diagram on the next page.

> **Given two straight lines, there are 3 possibilities:**
> A. The two lines intersect at one point.
> B. The two lines are parallel and don't intersect.
> C. The two lines are actually the same line, so their intersection is all the points on that line.



We just saw an example of case A. The following examples illustrate cases B and case C respectively.

E.g. #2. To find the intersection of $x - 3y = 4$ and $3y - x = 5$, we need to solve the system:
$$\begin{cases} x - 3y = 4 \\ 3y - x = 5 \end{cases}$$
We can do this by solving for x or for y in one of the equations and substituting into the other one. For example, we can solve for x in the first equation:
$x = 3y + 4$, and substitute this into the second:
$3y - (3y + 4) = 5$
$4 = 5$
Now, this is false, since $4 \neq 5$. This tells us that the system has no solutions, since if we assume both equations hold simultaenously we get a contradiction (that $4 = 5$). So this means the two lines don't intersect, which is because they're parallel.



E.g. #3. To find the intersection of $3x + 2y = 3$ and $9x + 6y = 9$, we need to solve the system:
$\begin{cases} 3x + 2y = 3 \\ 9x + 6y = 9 \end{cases}$
We can solve this by solving for x or for y in one of the equations and then substituting. From the first equation, we get:
$3x = 3 - 2y$
$x = 1 - \frac{2}{3}y$
Substituting into the second, we get:
$9(1 - \frac{2}{3}y) + 6y = 9$
$9 - 6y + 6y = 9$
$0 = 0$
This happens because the second equation is just a multiple of the first equation, so any point (x, y) which satisfies one will also satisfy the other. In effect, this means the two equations represent the same line. This can be seen by writing them in the form $y = mx + b$: they both give $y = -\frac{3}{2}x + \frac{3}{2}$. So the solution is the entire line: all points of the form $(x, -\frac{3}{2}x + \frac{3}{2})$ with $x \in \mathbb{R}$.

E.g. #4. Find the intersection of $y = 5$ and $3x - 2y = 8$.
As before, we have to solve the two equations simultaneously. We already have, in the first one, what y is, so we can substitute this into the second:
$3x - 2(5) = 8$
$3x = 18$
$x = 6$.
So the intersection is $(6, 5)$.

E.g. #5. Find the intersection of $x - 4y = -3$ and $6x + 2y = 5$.
We can solve for x using the first equation:
$x = 4y - 3$, and substitute this into the second:
$6(4y - 3) + 2y = 5$
$24y - 18 + 2y = 5$
$26y = 23$
$y = \frac{23}{26}$
Then we substitute this back to find x:
$x = 4(\frac{23}{26}) - 3 = \frac{46}{13} - 3 = \frac{7}{13}$.
So the point of intersection is $(\frac{7}{13}, \frac{23}{26})$. We can check our answer by substituting this into our equations and seeing whether they come out true.

Practice problems

For each of the following pairs of equations of lines, find their intersection and then draw a fully-labelled diagram representing the lines and the intersection.

#1. $y = -\frac{1}{2}x + 3$ and $x + y = 4$.

#2. $2x - 3y = 8$ and $x = -2$.

#3. $y = -4x + 2$ and $8x + 2y = -1$.

Homework

For each of the following pairs of equations of lines, find their intersection and then draw a fully-labelled diagram representing the lines and the intersection.

#1. $2x + y = 0$ and $x + 3y = -5$.

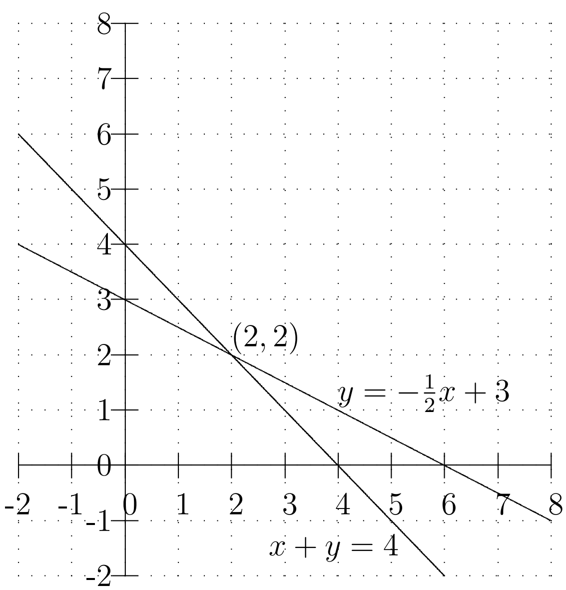
#2. $y = 3$ and $-x + y = -1$.

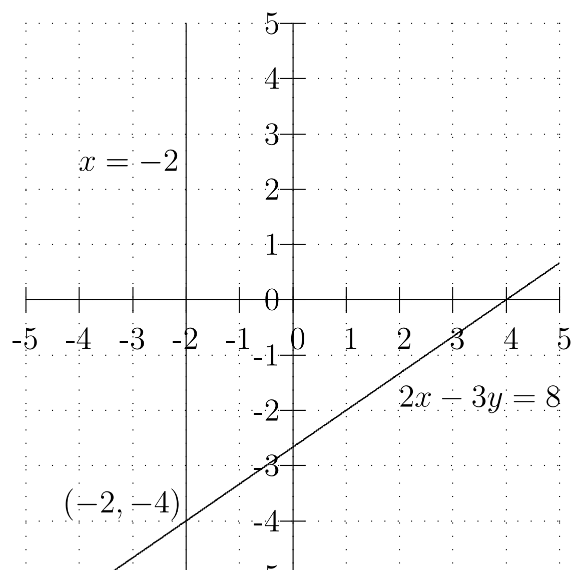
#3. $x + 3y = 3$ and $2x - 3y = 6$.

Solutions for practice problems

#1. $y = -\frac{1}{2}x + 3$ and $x + y = 4$.
Substituting the first into the second:
$x + (-\frac{1}{2}x + 3) = 4$
$\frac{1}{2}x = 1$
$x = 2$
Substituting this into the second:
$2 + y = 4$
$y = 2$.
So the point of intersection is $(2, 2)$.
For the diagram we also need the intersections with the axes. The first line intersects them at $(6, 0)$ and $(0, 3)$; the second at $(4, 0)$ and $(0, 4)$.

#2. $2x - 3y = 8$ and $x = -2$.
Substituting the second into the first:
$2(-2) - 3y = 8$
$-4 - 8 = 3y$
$-12 = 3y$
$-4 = y$
So the intersection is $(-2, -4)$.
For the diagram we also need the intersections with the axes. The first line intersects them at $(4, 0)$ and $(0, -\frac{8}{3})$; the second is a vertical line at $x = -2$.





#3. $y = -4x + 2$ and $8x + 2y = -1$.
Substituting the first equation into the second:
$8x + 2(-4x + 2) = -1$
$8x - 8x + 4 = -1$
$4 = -1$
This is false, so there are no solutions, meaning the lines are parallel.
For the diagram we need the intersections with the axes. The first line intersects them at $(\frac{1}{2}, 0)$ and $(0, 2)$; the second at $(-\frac{1}{8}, 0)$ and $(0, -\frac{1}{2})$