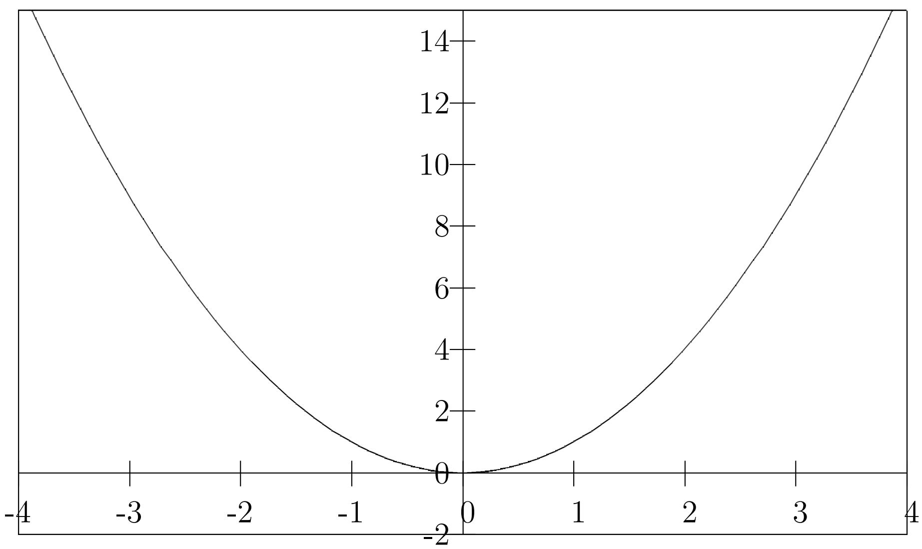
Functions: range, monotonicity, curvature
Mr. Neeman, 10A. October 3, 2011

Range

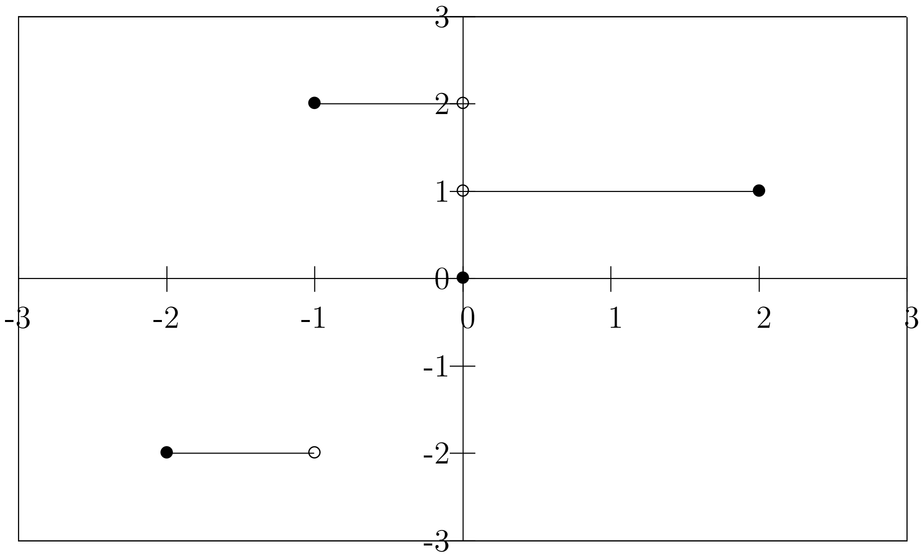
Definition: The **range** of a function is the set of all images. An equivalent definition is that the range is the set of all those elements of the codomain which have at least one preimage.

To find the range, the easiest way is using the function's graph. One has to look at which values of y the function takes, and the set of those values will be the range. A graphical way to think about it is that one is projecting the function's graph onto the y axis.
E.g. $f : \mathbb{R} \to \mathbb{R}$
$f(x) = x^2$
The graph is shown below:



From the graph, we can see that the range is $[0, \infty[$, because the function comes down from infinity, down to zero, and then goes back up to infinity, so it goes through all positive numbers and 0.

E.g. For the step function show below, the range is $\{-2, 0, 1, 2\}$ because there are the values of y which the function takes.



The range is always a subset of the codomain. It might be the entire codomain or it might not. An example of a function whose range is the same as the codomain is $f(x) = x$, since every real number has a preimage under this function.

Monotonicity (increasing, decreasing, constant)

With some functions, y increases as x increases. For example, $f(x) = 2x$ is such a function. With others, y decreases as x increases, for example $f(x) = -x$. For other functions, y is constant (e.g. $f(x) = 3$). But for some functions, whether y increases or decreases (or neither) as x increases depends on the value of x. For example, $f(x) = x^2$, shown in the graph on the previous page, goes up on the right side (positive x) and down on the left side (negative x). So when considering the increasing and decreasing of functions, we will first look at this behavior on intervals.

Definitions (intervals of monotonicity):
1. We say a function is **increasing on an interval** I (which must be a subset of the domain), if for any points x and y in I such that $y > x$, we have $f(y) \geq f(x)$. A better name for this would be **non-decreasing**, since this definition includes cases where the function is constant on the interval or parts of it. But this is how the term 'increasing' is used.
2. We say a function is **decreasing on an interval** I (which must be a subset of the domain), if for any points x and y in I such that $y > x$, we have $f(y) \leq f(x)$. A better name for this would be **non-increasing**, since this definition includes cases where the function is constant on the interval or parts of it. But this is how the term 'decreasing' is used.
3. We say a function is **strictly increasing on an interval** I (which must be a subset of the domain), if for any points x and y in I such that $y > x$, we have $f(y) > f(x)$.
4. We say a function is **strictly decreasing on an interval** I (which must be a subset of the domain), if for any points x and y in I such that $y > x$, we have $f(y) < f(x)$.

Definitions (monotonicity of a function):
1. We say a function is **increasing**, if for any points x and y in its domain such that $y > x$, we have $f(y) \geq f(x)$. A better name for this would be **non-decreasing**, since this definition includes cases where the function is constant on some or the whole of the domain. But this is how the term 'increasing' is used.
2. We say a function is **decreasing**, if for any points x and y in its domain such that $y > x$, we have $f(y) \leq f(x)$. A better name for this would be **non-increasing**, since this definition includes cases where the function is constant on some or the whole of the domain. But this is how the term 'increasing' is used.
3. We say a function is **strictly increasing**, if for any points x and y in its domain such that $y > x$, we have $f(y) > f(x)$.
4. We say a function is **strictly decreasing**, if for any points x and y in its domain such that $y > x$, we have $f(y) < f(x)$.

E.g. The function $f(x) = x^2$ is increasing on $[0, \infty[$ and decreasing on $]-\infty, 0]$. It is also strictly increasing and strictly decreasing on these, respectively.

E.g. The function $f(x) = 2x$ is increasing (also strictly increasing).

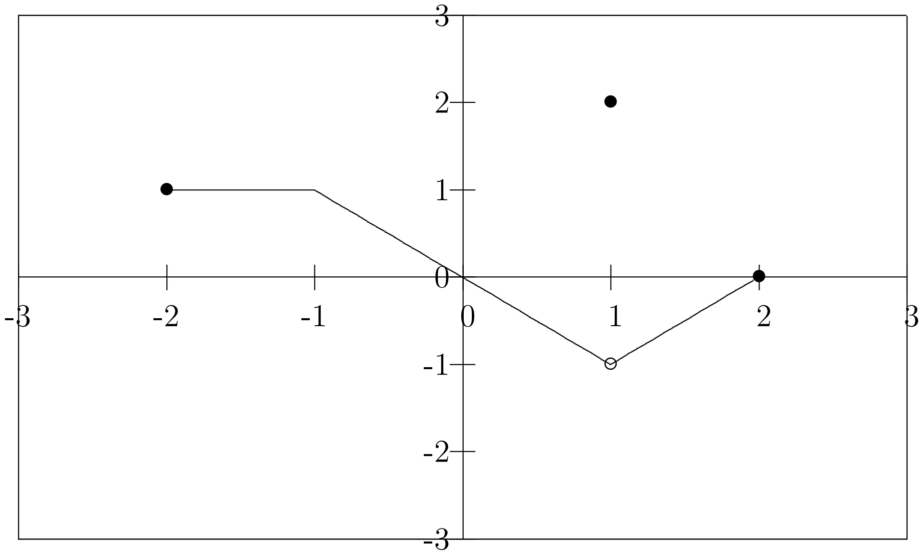
Curvature

Curvature is about which way a graph is turning. A line can be either flat (a straight line) or it can be curving up or down. If its curving up, we say it's concave up, or that it has positive curvature. If it's curving down, we say it's concave down, or has negative curvature.

E.g. $f(x) = x^2$ has positive curvature. $f(x) = -x^2$ has negative curvature.

Practice exercises

#1. Consider the following graph:



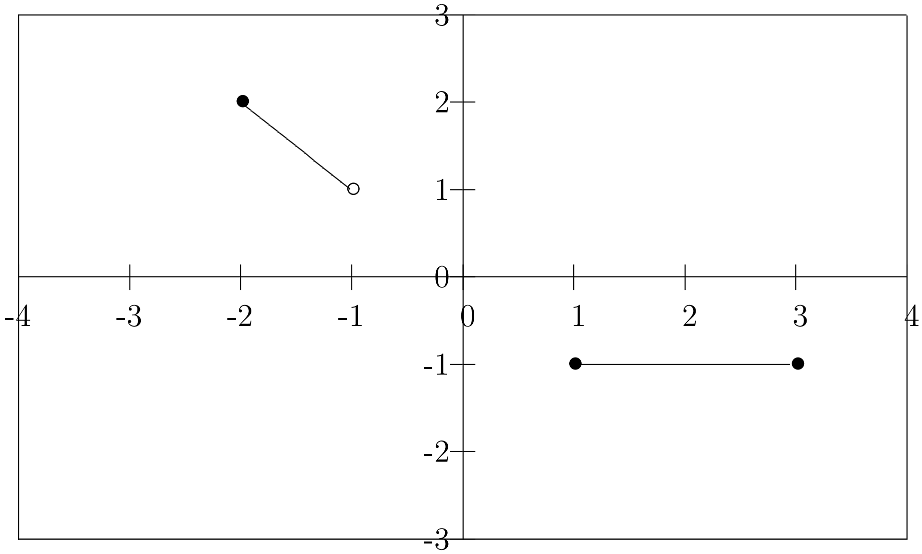
(a) Find the image of -1.
(b) Find any preimages of -1.
(c) Find the range.
(d) Find any intervals where the function is strictly decreasing.

#2. Consider the function $f(x) = 1 - x^2$ (with domain and codomain $\mathbb{R}$).
(a) Sketch the function's graph, using a table of values.
(b) Find the image of -3.
(c) Find any preimages of 0.
(d) Find the function's range.
(e) Find the function's intervals of monotonicity (i.e. find where the function is increasing and where it's decreasing).
(f) What is the function's concavity?

Homework (for Friday, Oct. 7th)

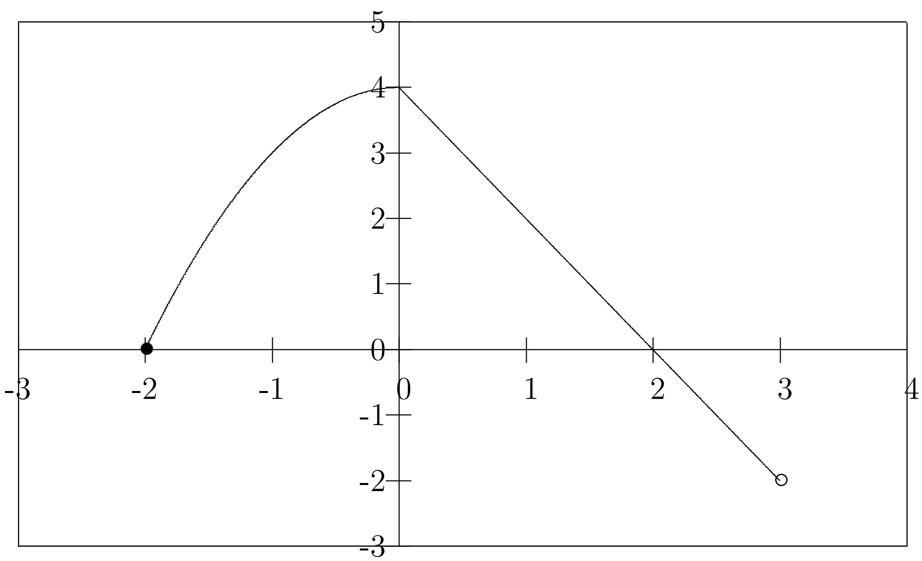
#1. Consider the function $f(x) = 1 + \sqrt{x}$, with domain $[0, \infty[$ and codomain $\mathbb{R}$.
(a) Find the image of 7.
(b) Find any preimages of 5
(c) Find any preimages of 0.
(d) Sketch the function's graph, using a table of values.
(e) What is the function's range?
(f) Is the function increasing or decreasing?
(g) What is the function's concavity?

#2. Consider the function whose graph is shown below.



(a) What is the function's domain?
(b) What is the function's range?
(c) Where is the function strictly decreasing?
(d) Find the image of 2.
(e) Find any preimages of 2.

#3. Consider the function whose graph is shown below.

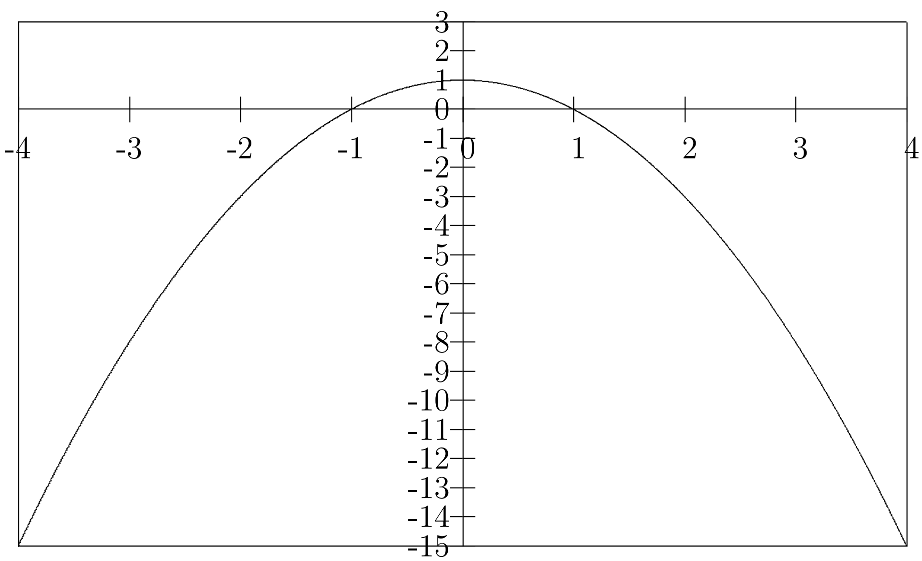


(a) Find the image of 0.
(b) Find any preimages of 0.
(c) What is the function's domain?
(d) What is the function's range?
(e) Where is the function increasing and where is it decreasing?

Solutions for practice exercises

#1.
(a) Find the image of -1.
1
(b) Find any preimages of -1.
There aren't any (since the image of 1 is 2, not -1).
(c) Find the range.
$]-1,1]\cup\{2\}$
(d) Find any intervals where the function is strictly decreasing.
$[-1,1[$

#2. Consider the function $f(x)=1-x^2$ (with domain and codomain $\mathbb{R}$).
(a) Sketch the function's graph, using a table of values.
Here is the graph (I didn't use a table of values, but it will come out the same).



(b) Find the image of -3.
$f(-3)=1-(-3)^2=1-9=-8$
(c) Find any preimages of 0.
We set $f(x)=0$, which gives $1-x^2=0$, so that $x^2=1$. Solving this, we get $x=1$ and $x=-1$, so these are the two preimages. You can also use the graph (or the table of values if it contains these values).
(d) Find the function's range.
$]-\infty,1]$
(e) Find the function's intervals of monotonicity (i.e. find where the function is increasing and where it's decreasing).
The function is increasing on $]-\infty,0]$ and decreasing on $[0,\infty[$.
(f) What is the function's concavity?
Concave down.