Radical equations

Mr. Neeman. 10A, August 12, 2011

In math, a radical is a root of some sort (square root, cubic root, etc.). A **radical equation** is an equation in which the unknown (usually x) appears inside a radical. For example: $\sqrt{x} = 2$ would be considered a radical equation. To solve this one, we just square both sides, and we get $x = 4$. We can check and see that 4 is, in fact, a solution since $\sqrt{4} = 2$.

We'll be solving radical equations which are more complicated than this. However, the general strategy we'll use will be the same: **we will isolate the radical on one side of the equation, and then raise both sides of the equation to the appropriate power to get rid of the root (e.g. we will square both sides if it's a square root, cube if it's a cubic root, etc.).** If there is more than one radical in our equation, we might have to first eliminate one and then eliminate the other (or others).

One thing to watch out for is **extraneous solutions**. This is a misnomer, as they're not actually solutions to the equation. The following example illustrates this:

E.g. $\sqrt{2x+3} = x$

$\sqrt{2x+3}^2 = x^2$

$2x + 3 = x^2$

$0 = x^2 - 2x - 3$

$0 = (x-3)(x+1)$

Now, this last equation has solutions $x = 3$ and $x = -1$. However, if you then try to substitute these into the original equation, you will find that $x = 3$ is a solution, but $x = -1$ isn't. So the solution to our original equation is just $x = 3$. $x = -1$ is called an **extraneous solution**. It is a value for x which arises in working from the equation, but which is not really a solution. For this reason, **it's essential to test all of one's solutions whenever solving a radical equation**.

Here's a simpler example illustrating extraneous solutions:

E.g. $\sqrt{x} = -x$. Squaring both sides, we get:

$x = x^2$, which has solutions $x = 0$ and $x = 1$. However, $x = 1$ is not a solution for the original equation, so it's an extraneous solution. Therefore, our original equation's only solution is $x = 0$.

Here is a slightly more complicated example:

E.g. $\sqrt{x^2+3} = x + 1$

$x^2 + 3 = (x+1)^2$

$x^2 + 3 = x^2 + 2x + 1$

$2 = 2x$

$1 = x$

We then verify that $x = 1$ satisfies the original equation, which it does, so this is our solution.

Note that, depending on the original equation you could end up with different kinds of equations after eliminating the radicals: you can get a linear equation as in the last example, a quadratic equation as in the previous ones, etc.

Exercises: solve the following equations.

#1. $\sqrt{2x+1}+1=x$

#2. $\sqrt{4x-2}+5=0$

#3. $\sqrt[3]{5x-3}=2$

#4. $\sqrt{2x}-1=\sqrt{x+1}$

#5. $\sqrt{x+2}-\sqrt{x-3}=1$

#6. $\sqrt{x+2}-\sqrt{2x+2}=-1$

#7. $\sqrt{x^2+x+1}=\sqrt{3+x}$

#8. $x=\sqrt{x-x^2-x^3-1}-1$

#8. $x=\sqrt{x-x^2-x^3-1}-1$

#9. $\sqrt{x^3+16x^2+28x+14}-3x-2=0$

#10. $\sqrt{3x^2+9x+1}-2=x$

Solutions.

#1. $\sqrt{2x+1}+1=x$
$\sqrt{2x+1}=x-1$
$2x+1=(x-1)^2$
$2x+1=x^2-2x+1$
$0=x^2-4x$
$0=x(x-4)$
Therefore $x=0$ or $x=4$. However, we have to test each of these. Doing that, we find that $x=0$ is extraneous. So the solution is $x=4$.

#2. $\sqrt{4x-2}+5=0$
$\sqrt{4x-2}=-5$
Here we can see that this has no solutions, since $\sqrt{4x-2}$ can never be negative. However, if you don't notice that, you can square both sides:
$4x-2=25$
$x=\frac{27}{4}$
If you then test this solution you will find it's extraneous, which means there are no solutions.

#3. $\sqrt[3]{5x-3}=2$
$5x-3=8$
$x=\frac{11}{5}$
Testing this solution, we find that it does work, so the answer is $x=\frac{11}{5}$.

#4. $\sqrt{2x}-1=\sqrt{x+1}$
Here we have two radicals. We have to choose which one to eliminate first. Let's say we want to eliminate $\sqrt{x+1}$:
$2x-2\sqrt{2x}+1=x+1$

Now we have to eliminate the remaining radical:
$2\sqrt{2x} = x$
$4(2x) = x^2$
$x^2 - 8x = 0$
$x(x-8) = 0$
So $x = 0$ or $x = 8$. Testing these, we find that $x = 0$ is extraneous, and the answer is $x = 8$.

#5. $\sqrt{x+2} - \sqrt{x-3} = 1$
This one is similar, we have to first eliminate one radical, and then eliminate the remaining radical.
$\sqrt{x+2} = 1 + \sqrt{x-3}$
$x + 2 = 1 + (x-3) + 2\sqrt{x-3}$
$2\sqrt{x-3} = 4$
$\sqrt{x-3} = 2$
$x - 3 = 4$
$x = 7$
Testing this, we find that it works, so the answer is $x = 7$.

#6. $\sqrt{x+2} - \sqrt{2x+2} = -1$
Again, we eliminate one first.
$\sqrt{x+2} = -1 + \sqrt{2x+2}$
$x + 2 = 1 + (2x+2) - 2\sqrt{2x+2}$
$2\sqrt{2x+2} = x + 1$
$4(2x+2) = x^2 + 2x + 1$
$x^2 - 6x - 7 = 0$
$(x-7)(x+1) = 0$
So either $x = -1$ or $x = 7$. Testing these out, we find that $x = -1$ is extraneous, so our answer is $x = 7$.

#7. $\sqrt{x^2+x+1} = \sqrt{3+x}$
Here we eliminate both radicals together:
$x^2 + x + 1 = 3 + x$
$x = \pm\sqrt{2}$
Testing them, we find they both work, so our answer is $x \in \{-\sqrt{2}, \sqrt{2}\}$

#8. $x = \sqrt{x - x^2 - x^3 - 1} - 1$
$x + 1 = \sqrt{x - x^2 - x^3 - 1}$
$x^2 + 2x + 1 = x - x^2 - x^3 - 1$
$x^3 + 2x^2 + x + 2 = 0$
We solve this as before by testing the divisors of 2, and we get:
$(x+2)(x^2+1) = 0$
So $x = -2$.
However, when we check this answer we find it's extraneous, so there are no solutions.

#8. $2x - \sqrt{3x^2+2x+2} + 1 = 0$
$2x + 1 = \sqrt{3x^2+2x+2}$
$4x^2 + 4x + 1 = 3x^2 + 2x + 2$
$x^2 + 2x - 1 = 0$

$(x+1+\sqrt{2})(x+1-\sqrt{2})=0$
So $x=-1-\sqrt{2}$ or $x=-1+\sqrt{2}$.
Testing them shows the first is extraneous, so our answer is $x=-1+\sqrt{2}$.

#9. $\sqrt{x^3+16x^2+28x+14}-3x-2=$
$\sqrt{x^3+16x^2+28x+14}=3x+2$
$x^3+16x^2+28x+14=9x^2+12x+4$
$x^3+7x^2+16x+10=0$
$(x+1)(x^2+6x+10)=0$
The remaining quadratic has no solutions, so the only solution is $x=-1$. Testing it, we find that it's extraneous, so there are no solutions.

#10. $\sqrt{3x^2+9x+1}-2=x$
$\sqrt{3x^2+9x+1}=x+2$
$3x^2+9x+1=x^2+4x+4$
$2x^2+5x-3=0$
$(2x-1)(x+3)=0$
So $x=\frac{1}{2}$ or $x=-3$. Testing these, we find $x=-3$ is extraneous, so our answer is $x=\frac{1}{2}$.