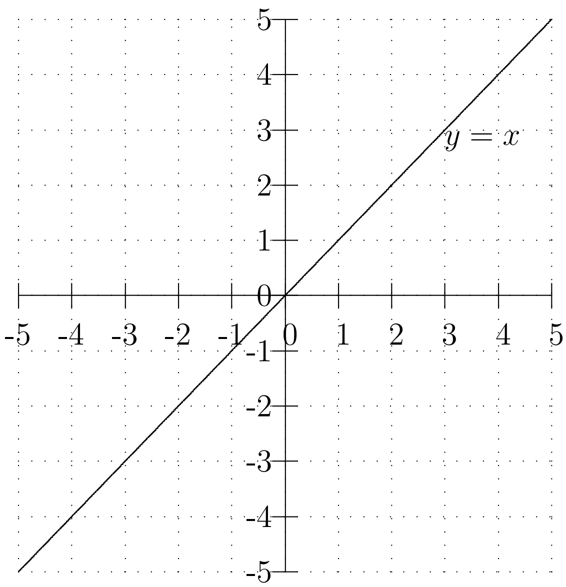
Linear functions and analytic geometry: introduction
Mr. Neeman, 10A. November 1, 2011

A linear function is one of the form $f(x) = mx + b$, where m and b are constants.

Examples:
$f(x) = x$
$f(x) = 3x - 1$
$f(x) = 6 - 2x$, etc.

The simplest linear functions are ones we've already studied:
When $m = 0$, the linear function is simply a constant function: $f(x) = b$. Its graph is just a horizontal line at height b.

After constant functions, the simplest linear function is $f(x) = x$.
This is just a function which, given any input, gives that same number as an output. Its graph has the equation $y = x$, and contains points whose x and y coordinates are equal to each other:

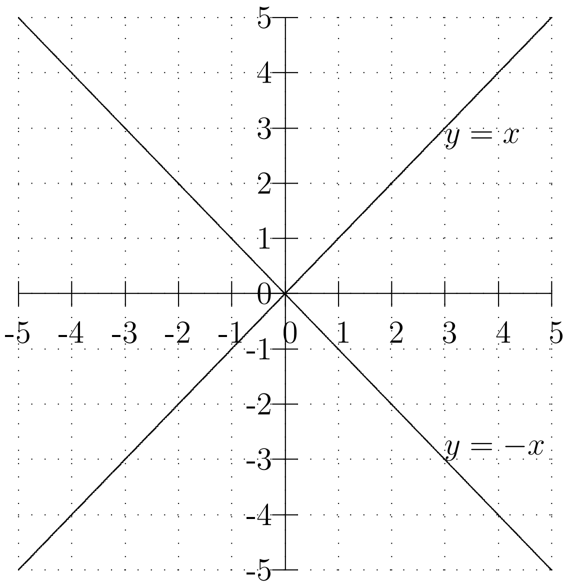


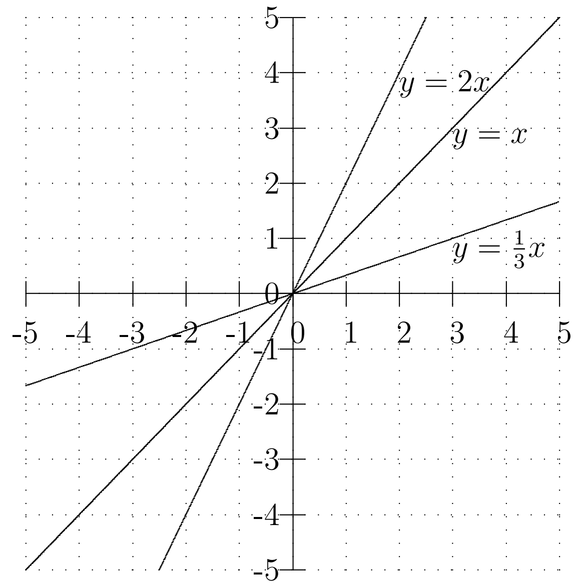
This is a straight line. In fact, the graph of any linear function is a straight line, which is where the name comes from.

Every other linear function, can be seen to be a transformation of this one:
Suppose $f(x) = x$, and m and b are constants.
Then $mf(x) + b = mx + b$.
This means that the graph of the function $g(x) = mx + b$ will be the graph of $f(x) = x$ scaled vertically by a factor of m (if m is negative that includes a reflection), and then translated vertically by b units.
Let's first look at each of these separately.

Reflection
Suppose $g(x) = -x$. Taking, as before, $f(x) = x$, we know that the graph of $g(x)$ will simply be that of $f(x)$ reflected vertically around the y axis. This is shown in the diagram on the left below.

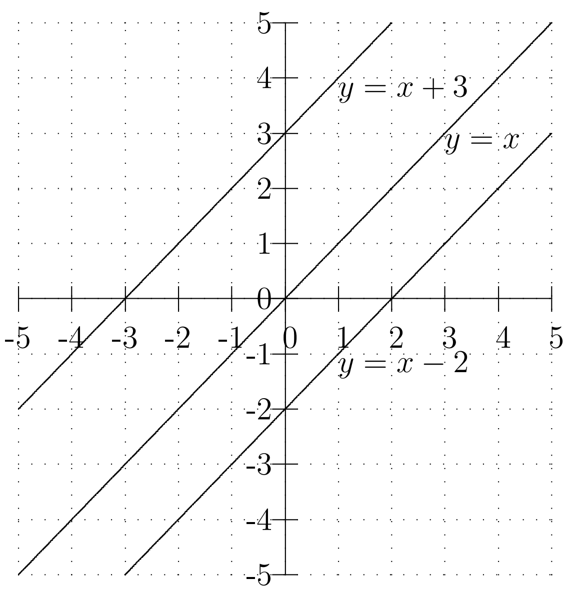
Scalings
Suppose $g(x) = mx$. Taking, as before, $f(x) = x$, we know that the graph of $g(x)$ will simply be that of $f(x)$ scaled vertically by a factor of m. Several examples of this are shown in the diagram on the right below.

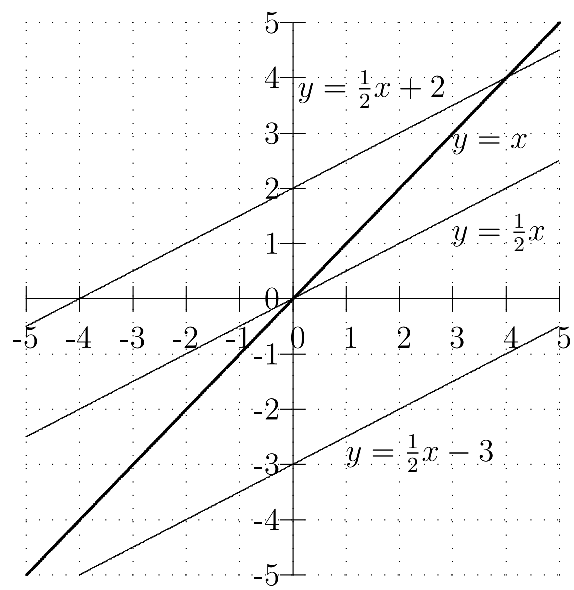




Notice that, in all these cases, the line goes through the origin.

Translations
Suppose $g(x) = x + b$. This will be the graph of $f(x) = x$, translated by b units vertically. The diagram on the left below shows the examples of $b = 3$ and $b = 2$.
Likewise, $y = \frac{1}{2}x + b$ will be the graph $y = \frac{1}{2}x$ translated vertically by b units. The diagram on the right below show some examples, as well as $y = x$ (for comparison).





Gradient, slope, and y-intercept

You will have noticed that multiplying $f(x) = x$ by a constant, m, changes the graph's slope. If $m > 1$, the line becomes steeper and if $0 < m < 1$, the line becomes less steep. If $m = 1$, of course, there's no change, since multiplying anything by 1 leaves it the same. And if $m = 0$, of course, we just get $g(x) = 0$, which is a constant function.

m is called the function's, or the line's, **gradient**. It is sometimes also called the **slope**. However, these two are not really the same: the gradient can be negative, whereas the slope cannot. This is like the relationship between distance and displacement which we encountered in the context of projectile motion.

So, for example, $f(x) = -2x$ has gradient -2, and slope 2. In spanish, the word "pendiente" is used to mean "gradient," and there isn't (as far as I know) a separate word for "slope." For this class we won't worry about slopes, we'll just work with the gradient instead.

The gradient is defined by the following formula:

$$\text{gradient} = m = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Where (x_1, y_1) and (x_2, y_2) are two points on the line.

For example, looking at the graph of $y = 2x$ on the previous page, we see it goes through the origin, $(0, 0)$, and through $(4, 2)$. This means its gradient is $\frac{4-0}{2-0} = 2$, which is right.

A linear function is strictly increasing if $m > 0$, constant if $m = 0$, and strictly decreasing if $m < 0$.

We can summarize, then:

The graph of $g(x) = mx$ is a straight line, with gradient m, passing through the origin.

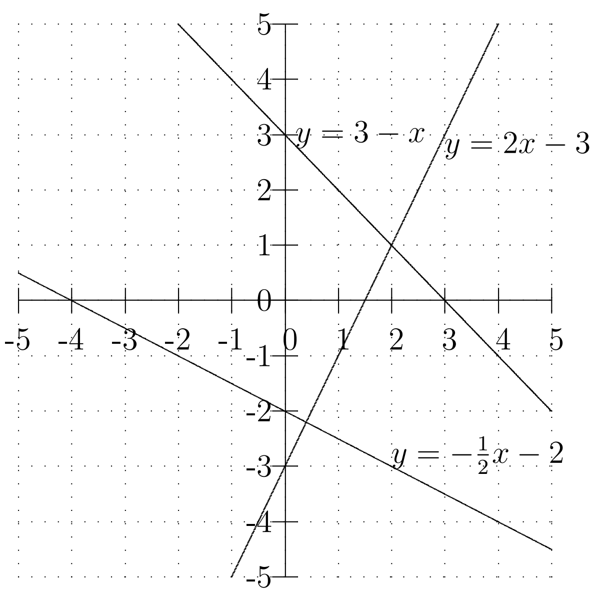
If we then add the constant b, that is a vertical translation, so that the line will no longer pass through the origin. Instead it will pass through the point $(0, b)$. We can see this is so by substituting $x = 0$:

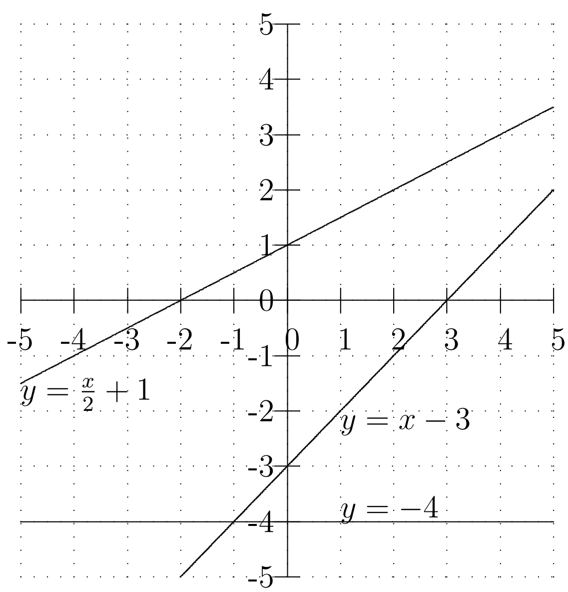
If $g(x) = mx + b$, then the intersection with the y axis (called the y-**intercept**), will be when $y = g(0) = 0x + b = b$.

Therefore:

The graph of $g(x) = mx + b$ is a straight line with gradient m and passing through the point $(0, b)$.

The diagrams below show some examples.





Finding the intersections with the axes

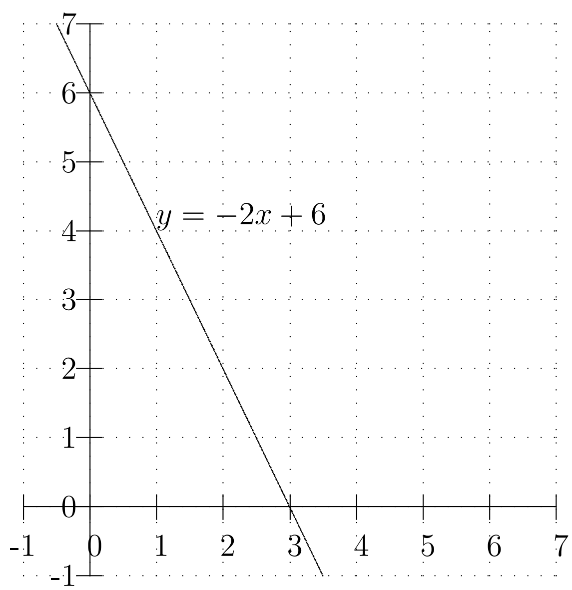
Recall that to find the intersection with the y axis, we set $x = 0$ and find y. And to find the intersection with the x axis, we set $y = 0$, and find the value of x.

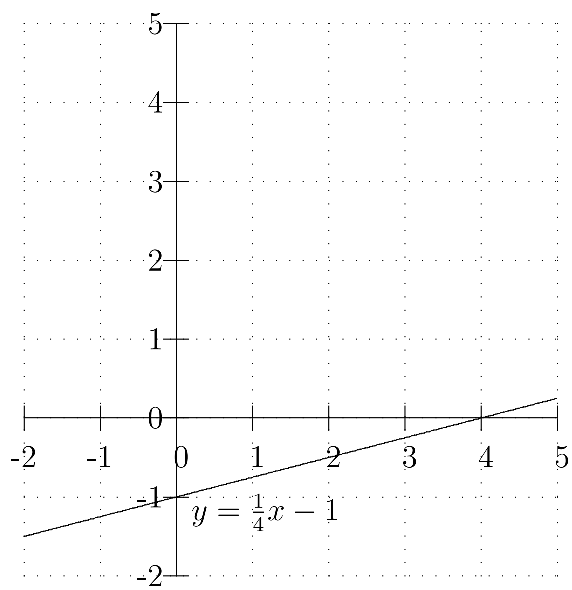
Sketching graphs

Sketching the graph of a linear function is easy, because it's a straight line. Given any two points, there's exactly one straight line that goes through them, which is easy to draw with a ruler. So to sketch a linear function, you just need a table of values for two values of x. Taking $x = 0$ as one of them makes it very simple, since if $f(x) = mx + b$, then $f(0) = b$ (i.e. as seen before, the graph intersects the y-axis at $(0, b)$).
Since we generally want our sketches to include the intersections with the axes, it's a good idea to simply use those as our two points. So we just find the intersections with the x and y axes, and draw a straight line between them. **Of course, if the line goes through the origin, it will intersect both axes at that one point, so we need to use a different value of x for the second point**.

E.g. #1. Sketch the graph of $f(x) = -2x + 6$.
To find the intersection with the y axis, we set $x = 0$, and find $y = f(0) = 6$.
To find the intersection with the x axis, we set $y = 0$, so $-2x + 6 = 0$, which we solve to get $x = 3$.
So our two points are $(0, 6)$ and $(3, 0)$, and the graph is as shown below on the left.

E.g. #2. Sketch the graph of $f(x) = \frac{1}{4}x - 1$.
To find the intersection with the y axis, we set $x = 0$, and find $y = f(0) = -1$.
To find the intersection with the x axis, we set $y = 0$, so $\frac{1}{4}x - 1 = 0$, which we solve to get $x = 4$.
So our two points are $(0, -1)$ and $(4, 0)$, and the graph is as shown below on the right.





Homework
For each of the following functions, find the gradient, and sketch the graph, labeling the intersections with the axes.
#H1. $f(x) = x - \frac{1}{2}$
#H2. $f(x) = 3 + 2x$
#H3. $f(x) = -\frac{2}{3}x + 1$
#H4. $f(x) = 2$