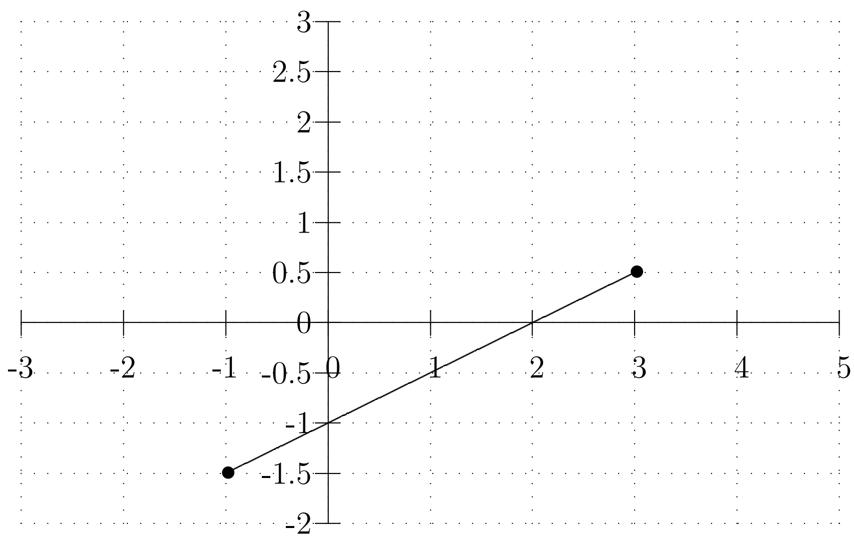
Linear functions with non-maximal domains
Mr. Neeman, 10A. November 17, 2011

Unless stated otherwise, we assume a function's domain is its maximal domain. Since a linear function never involves anything like a division by zero or a square root of a negative number, its maximal domain is $\mathbb{R}$. However, one can also consider linear functions with domains which are smaller than $\mathbb{R}$. We will be looking at **ones whose domains are intervals**. Essentially, this means we're just looking at part of the straight line, rather than the entire straight line.

E.g. Consider the function $f : [-1, 3] \to \mathbb{R}$, with $f(x) = \frac{1}{2}x - 1$. Its graph is like $y = \frac{1}{2}x - 1$, which we know how to draw, except that it only goes between $x = -1$ and $x = 3$. Remember that to sketch a line we needed two points. We usually used the intersections with the axes. However, if we're only considering part of the line for our function, it might not intersection one or any of the axes. If our domain is a finite interval, to sketch the function's graph we just take its endpoints as our two points:
When $x = -1$, we get $y = f(-1) = \frac{1}{2}(-1) - 1 = -\frac{3}{2}$, so the point is $(-1, -\frac{3}{2})$
When $x = 3$, we get $y = f(3) = \frac{1}{2}(3) - 1 = \frac{1}{2}$, so the point is $(3, \frac{1}{2})$.
Therefore, the graph is just the line segment between $(-1, -\frac{3}{2})$ and $(3, \frac{1}{2})$ (the endpoints are included since our domain includes its endpoints. As usual we want to label the intersections with the axes, which we find the usual way.



We note that this function is injective, but not surjective (whereas if it had domain $\mathbb{R}$ it would be surjective as well). Its range is $[-\frac{3}{2}, \frac{1}{2}]$.

We can do a lot of this analysis without sketching the graph. We just need to keep in mind that linear function map intervals onto intervals. So, if we have to domain, all we need to find the range is take the endpoints.

E.g. Consider the function $f : [3, 7] \to \mathbb{R}$, with $f(x) = -x + 4$. Find its range.
To find the range we need to find the images of the domain's endpoints:
$f(3) = -(3) + 4 = 1$
$f(7) = -(7) + 4 = -3$
Therefore, the function starts at $f(x) = 1$ and goes down to $f(x) = -3$, so its range is the interval $[-3, 1]$. The endpoints are included because they were included in the domain.

E.g. Consider the function $f :]-2, \infty[\to \mathbb{R}$, with $f(x) = -2x - 1$. Find its range.
To find the range we need to find the images of the domain's endpoints:
$f(-2) = -2(-2) - 1 = 3$.
Note that, since -2 isn't included in the domain, $f(-2)$ is actually not defined, so what's written in the previous line is technically wrong. However, we'll consider this a 'sin of convenience' to avoid complicated how we write the calculation.
For the other endpoint, we can't find $f(x)$, since ∞ isn't a real number. But we know the function is decreasing, because the gradient is negative, so when x goes to infinity, $f(x)$ goes to negative infinity.
Therefore, the function's range is $]-\infty, 3[$.

Homework

#1. Consider the function $f : [-1, 3] \to \mathbb{R}$, with $f(x) = 3x - 2$
(a) Sketch the function's graph, labelling the endpoints and any intersections with the axes.
(b) Find the function's range.
(c) Is the function surjective?
(d) Is the function injective?
(e) Is the function bijective?

#2. Consider the function $f : [3, 8] \to \mathbb{R}$, with $f(x) = -x - 2$.
(a) Sketch the function's graph, labelling the endpoints and any intersections with the axes.
(b) Find the function's range.
(c) Is the function surjective?
(d) Is the function injective?
(e) Is the function bijective?