Application of quadratic equations: free-fall

Mr. Neeman. 10A, August 5, 2011

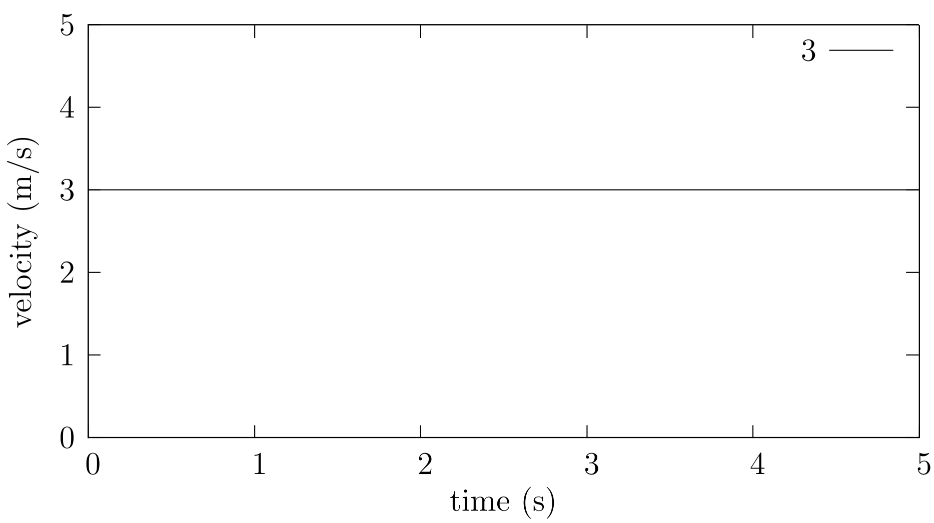
When we say "displacement", what we mean is the position of an object relative to some fixed point. For example, we can refer to vertical displacement, taking $y = 0$ to be ground-level, and using meters as our units. Then, for example, $y = 3$ would be 3 meters above the ground, and $y = -2$ would be two meters below the ground.

The difference between displacement is that distance is the absolute value of displacement (i.e. distance is never negative). So, for example, $y = -3$ is distance 2 from ground-level, but it's displacement relative to ground level is negative (i.e. downwards).

There is a parallel distinction between velocity and speed. Velocity and displacement have a direction, whereas distance and speed don't (you can have negative velocity, but not negative speed).

Consider an object moving upwards at a constant velocity of 3 meters per second (3 m/s), which is to say it moves up 3 meters every second. After 5 seconds, it will have moved 15 meters. Now, figure 1 shows the velocity:

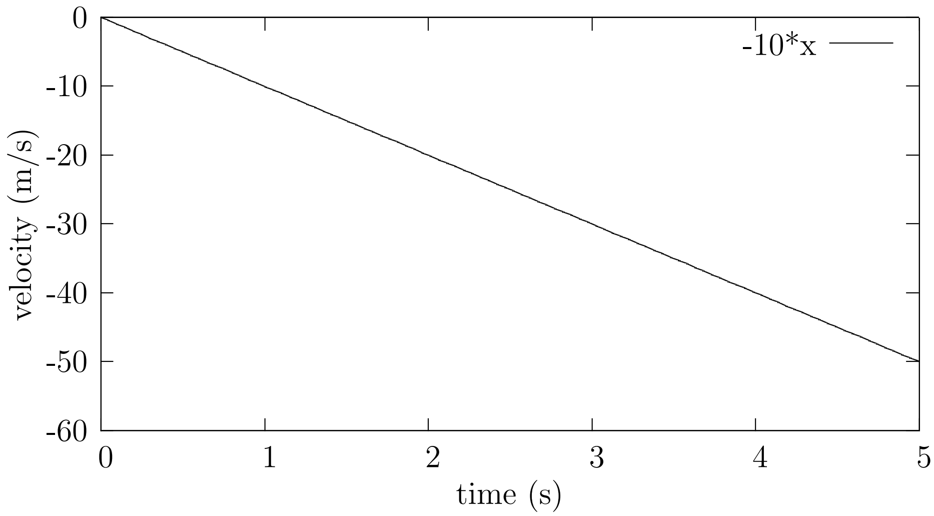
Figure 1: Motion at constant velocity



The key thing to notice is that the displacement, 15 meters, is also given as the area between the graph and the x-axis (i.e. the rectangle of length 5 seconds and width 3 m/s). This is true, in general: *the area between the graph of an object's velocity equals the object's displacement (with areas under the x-axis being counted as negative).*

We can now apply this to a case of free fall. Gravity effects a constant acceleration, which we can take to be approximately -10 m/s^2. What this means is that after each second of free-fall, the object's velocity will be 10 m/s less than before. If our object starts with velocity zero and enters free-fall, then at time t its velocity will be $-t$ m/s. For example, let's consider the case of free-fall lasting 5 seconds. This is shown in figure 2.

Figure 1: Motion at constant velocity



The area between the line can be calculated using the formula for the area of a triangle. Its base is 5 seconds, and it's height is 50 m/s. So the area is $\frac{1}{2} \times 5 \times 50 = 125$ meters. Since it's under the x-axis, the displacement is -125 meters.

Now, let's generalize to time t seconds. At the end of t seconds, the velocity would be $-10t$ m/s. So the height of the triangle would be $10t$ m/s. The base of the triangle would be the time elapsed, which is t seconds. Therefore, the area would be $\frac{1}{2} \times t \times 10t = 5t^2$. So the displacement will be $-5t^2$. Therefore, we can write $y = -5t^2$.

This was for a particle starting at $y = 0$ and with velocity zero. If instead we start at $y = y_0$ (read "y zero", the value of y at time zero), we would get $y = y_0 - 5t^2$. Additionally, if we start with an initial velocity of v_0, this would shift our velocity curve up by v_0, which would add v_0t to the area. So we get our final formula:

Displacement under free-fall: $y = y_0 + v_0t - 5t^2$, where y_0 is the initial vertical displacement relative to the ground, and v_0 is the initial velocity.

We can use this to solve problems about free-fall. For example:

E.g. #1. Suppose an object is thrown up, at time $t = 0$, with a velocity of 4 m/s, from a height of 1 meter. When will it hit the ground?

Using $y_0 = 5$ and $v_0 = 3$, we get the equation $y = 5 - 4t - 5t^2$. The object will hit the ground when it's at ground-level, i.e. when $y = 0$. So we need to solve $0 = 1 - 4t - 5t^2$:
$5t^2 + 4t - 1 = 0$
$(5t - 1)(t + 1) = 0$
Either $5t - 1 = 0$ or $(t + 1) = 0$. Therefore, $t = \frac{1}{5}$ or $t = -1$.
Now, we're assuming the object was thrown at $t = 0$, and considering what happens to it later, so we're only interested in the positive solution, $t = \frac{1}{5}$.

E.g. #2. Suppose you throw a ball from ground-level directly up at time $t = 0$ (if you're not told otherwise, you can always assume $t = 0$ at the beginning), and that it comes back to where you threw it after 2 seconds. Calculate (a) the velocity with which the ball was thrown, and (b) the maximum height it reaches.

(a) First, we take inventory of the information we have. One is that $y_0 = 0$ since the ball is thrown from ground-level. We don't know what v_0 is, that's what we have to calculate. So, for now, what we have is $y = v_0 t - 5t^2$. We also know that the ball is back at ground-level at $t = 2$. So $0 = 2v_0 - 5 \times 2^2 = 2v_0 - 20$. Therefore, we can solve for v_0 and get $v_0 = 10$ m/s. So the initial velocity was 10 m/s.
(b) Now, the maximum height is reached when the ball stops moving upwards and instead starts moving downwards. At that point in between, it's velocity is zero. So the question is what height the ball was at when its velocity was zero. But we know that its velocity at $t = 0$ was 10 m/s, and that it goes down by 10 m/s every second due to gravity. So that means it was zero at $t = 1$. So we have to calculate y when $t = 1$: $y = 10t - 5t^2 = 10 - 5 = 5$. So it reached a maximum height of 5 meters.

E.g. #3. Suppose an object is dropped from rest at a height of 150 meters, onto a building whose whose roof is 29 meters above the ground. How long will it take to hit the roof?

Here we know $v_0 = 0$ (because it's dropped from rest) and $y_0 = 150$, since that's the initial height. So $y = 150 - 5t^2$. We want to solve for $y = 25$ since that's the height of the roof.
$25 = 150 - 5t^2$
$5t^2 = 125$
$t^2 = 25$
$t = \pm 5$
But, as before, we're looking for positive times only, so $t = 5$. So it will hit the roof 5 seconds after being dropped.

<u>Exercises</u>

#E1. Suppose an object is thrown up, with an initial velocity of 5 m/s, from a height of 20 meters. Calculate (a) the time at which it reaches its maximum height, (b) the maximum height, and (c) the time at which it will hit the ground.
#E2. Suppose an object is dropped from rest and hits the ground $\frac{5}{2}$ seconds later. Calculate (a) the height from which it was dropped, and (b) how long it took to fall halfway to the ground.
#E3. Suppose an object is thrown downwards with a velocity of 15 m/s, from a height of 100 meters. Calculate how long it will take to hit the ground.
#E4. Suppose an object is thrown up from the ground with a velocity of 7 m/s in a room with a ceiling that is 2 meters high. Calculate when the ball will hit the ceiling.
#E5. Suppose an object is thrown up, with a velocity of 15 m/s, from a height of 20 meters. Calculate at what time will it (a) be back at a height of 20 meters, (b) at what time it will hit the ground.

#E1. Suppose an object is thrown up, with an initial velocity of 5 m/s, from a height of 20 meters. Calculate (a) the time at which it reaches its maximum height, (b) the maximum height, and (c) the time at which it will hit the ground.

(a) Since its initial velocity is 5 m/s, and it velocity will be decrease by 10 m/s every second (since it's in free-fall), it will take half a second for the velocity to reach zero, which is when it's at its maximum height (since it then starts going down).
(b) $y = 20 + 5t - 5t^2$. Since it reaches maximum height at $t = \frac{1}{2}$, we subtitute this into the equation: $y = 20 + \frac{5}{2} - \frac{5}{4} = 20 + \frac{5}{4} = 21.25$. So the maximum height is 21.25 meters.
(c) It hits the ground when $y = 0$. So we solve $20 + 5t - 5t^2 = 0$
$5t^2 - 5t - 20 = 0$
$t^2 - t - 4 = 0$
$t^2 - t + \frac{1}{4} - \frac{1}{4} - 4 = 0$
$(t - \frac{1}{2})^2 - \frac{17}{4} = 0$
$(t - \frac{1}{2})^2 = \frac{17}{4}$
$t = \frac{1}{2} \pm \frac{\sqrt{17}}{2}$
Now, one of these is negative (since $\sqrt{17} > 1$), so that's not what we want. So the answer is $\frac{1+\sqrt{17}}{2}$ seconds.

#E2. Suppose an object is dropped from rest and hits the ground $\frac{5}{2}$ seconds later. Calculate (a) the height from which it was dropped, and (b) how long it took to fall halfway to the ground.

(a) We know $v_0 = 0$, so the formula is $y = y_0 - 5t^2$. We know $y = 0$ when $t = \frac{5}{2}$. Substituting this in, we get $0 = y_0 - 5(\frac{5}{2})^2$. So $y_0 = \frac{125}{4} = 31.25$. So the object was dropped from a height of 31.25 meters.
(b) Halfway to the ground from $\frac{125}{4}$ is $\frac{125}{8}$. So we want to solve $\frac{125}{8} = \frac{125}{4} - 5t^2$.
$5t^2 = \frac{125}{8}$
$t^2 = \frac{25}{8}$
$t = \pm\frac{5}{2\sqrt{2}}$
As usual, we're only looking for the positive root (since the object is dropped at $t = 0$.), so it gets halfway to the ground after $\frac{5}{2\sqrt{2}}$ seconds.
#E3. Suppose an object is thrown downwards with a velocity of 15 m/s, from a height of 100 meters. Calculate how long it will take to hit the ground.

$y = 100 - 15t - 5t^2$. We want to know when $y = 0$, so we solve $0 = 100 - 15t - 5t^2$:
$t^2 + 3t - 20 = 0$
$t^2 + 3t + \frac{9}{4} - \frac{9}{4} - 20 = 0$
$(t + \frac{3}{2})^2 - \frac{89}{4} = 0$
$t + \frac{3}{2} = \pm\frac{\sqrt{89}}{2}$
$t = -\frac{3}{2} \pm \frac{\sqrt{89}}{2}$
As usual, we want the positive root (and it is positive, since $\sqrt{89} > 3$), so it hits the ground at $t = \frac{\sqrt{89}-3}{2}$.

#E4. Suppose an object is thrown up from the ground with a velocity of 7 m/s in a room with a ceiling that is 2 meters high. Calculate when the ball will hit the ceiling.

We know $y_0 = 0$, $v_0 = 7$. So $y = 7t - 5t^2$. We want to find out when $y = 2$. So we need to solve $2 = 7t - 5t^2$.
$5t^2 - 7t + 2 = 0$
$(5t - 2)(t - 1) = 0$
So either $t = \frac{2}{5}$ or $t - 1$. Now, how do we interpret this? Well, this is assuming free-fall. But, of course, when the ball hits the ceiling it's no longer in free fall (the ceiling will change its direction suddenly). So we're looking for the smaller solution: $t = \frac{2}{5}$ seconds. The other solution $t = 1$ represents when the ball would be back at a height of 2 meters if there were no ceiling there to bump into.

#E5. Suppose an object is thrown up, with a velocity of 15 m/s, from a height of 20 meters. Calculate at what time will it (a) be back at a height of 20 meters, (b) at what time it will hit the ground.

(a) $y = 20 + 15t - 5t^2$. We want to solve $y = 20$:
$20 = 20 + 15t - 5t^2$
$0 = 15t - 5t^2$
$0 = 3t - t^2$
$0 = t(3 - t)$
So the two solutions are $t = 0$ (when it's thrown up) and $t = 3$. So the second one, 3 seconds, is when it's *back* at a height of 20 meters.
(b) Now we want to solve $y = 0$
$0 = 20 + 15t - 5t^2$
$t^2 - 3t - 4 = 0$
$(t - 4)(t + 1) = 0$
So either $t = 4$ or $t = -1$. But, as usual, we want the positive answer (i.e. after the ball is thrown), so the answer is that it will hit the ground in 4 seconds.