Functions (the beginning)

General concepts

A **function** is a relation between elements of two sets. The first set, is called the function's **domain**, and the second set is called the function's **codomain**. The function assigns to each element of the domain a corresponding element in the domain. Put differently, the function's takes in elements of the domain and outputs for each one an element of the codomain.

Note: for it to be a function, it must assign **exactly one** element of the codomain to each element of the domain. In other words, the function cannot give two outputs for the same input, nor can it fail to give an output for some input.

> The notation for functions is shown in this example:
> $f : \mathbb{R} \to \mathbb{N}$
> This is read as "f is function from the $\mathbb{R}$ to $\mathbb{N}$", or
> "f is a function with domain $\mathbb{R}$ and codomain $\mathbb{N}$."
>
> f is the function's name, which we use to refer to it and distinguish from others in case we're dealing with more than one function.
>
> $\mathbb{R}$ is the domain.
> $\mathbb{N}$ is the codomain.

Of course, this is just an example. We could have many others:

E.g. $g : \mathbb{R} \to \mathbb{R}$
This is read as "g is a function from $\mathbb{R}$ to $\mathbb{R}$", or "g is a function with domain $\mathbb{R}$ and codomain $\mathbb{R}$".

E.g. $f : \mathbb{Z} \to \mathbb{N}$
This is read as "f is a function from $\mathbb{Z}$ to $\mathbb{N}$", or "f is a function with domain F and codomain $\mathbb{N}$".

Constant functions

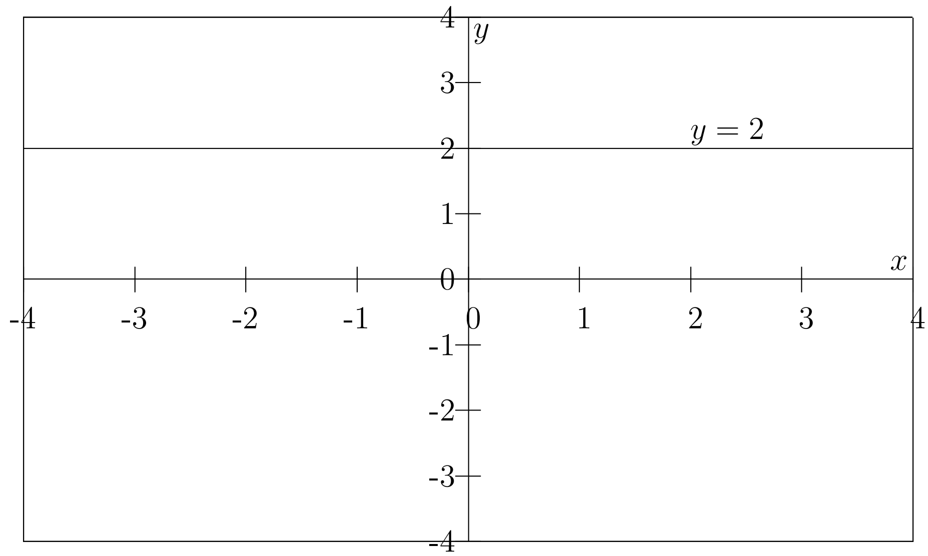
The simplest functions are ones whose output is the same regardless of what their input is. For example, we could have a function which takes in any real number, and outputs 2.

E.g.
$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto 2 \end{aligned}$$

Here, the first line is read as before, and the second is read as "f maps x to 2", "f takes x to 2", or x is mapped to 2".
Another way to express is to write:

$f : \mathbb{R} \to \mathbb{R}$
$f(x) = 2$

This, like all constant functions, is a very boring function. Whatever input you give it, it gives you as an output 2. Using rectangular coordinates, we can represent this function by means of a graph. The x coordinates will represent the input to the function, and the y coordinates will represent the value of the function, which is to say $f(x)$. Of course, for this function it means the y coordinate will always be 2, so the function's graph is a horizontal line at height 2:

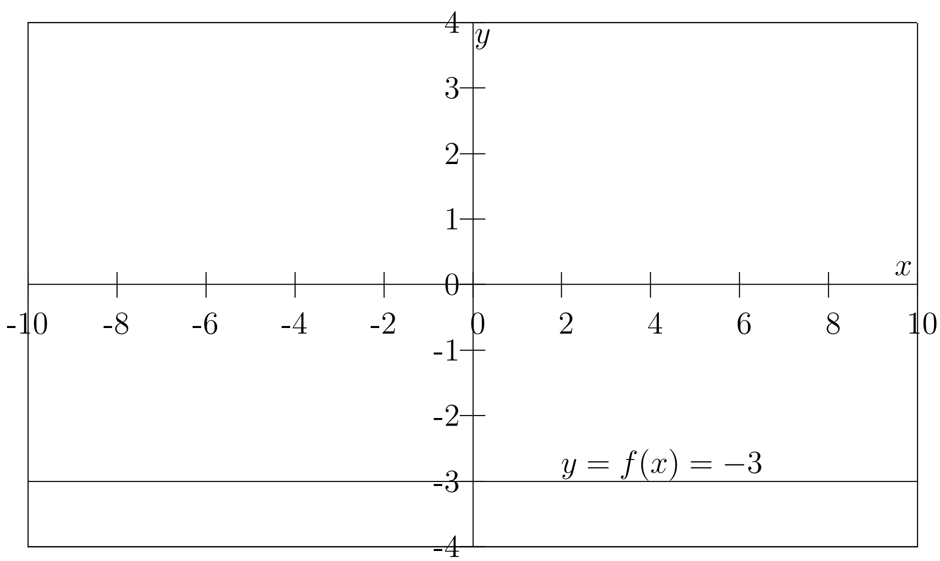


Of course, the function is defined on all real numbers, so this graph only shows part of it, from -4 to 4. The graph can be labeled either $y = f(x)$ or $y = 2$, since $f(x)$ equals 2 with this function. Which you do depends on what you want to emphasize (the function's name or its value). You could even label it $y = f(x) = 2$.

Another example:

E.g. $f : \mathbb{R} \to \mathbb{R}$
$\quad x \mapsto$ -3

This function, $f(x) = -3$, always gives as your output -3, regardless of what the input is. It's graph is:



Notice here I've plotted a bigger interval on the x axis. This is up to you to decide what's appropriate for each function. But in the case of constant functions it doesn't really matter, since it's a horizontal line all the way.

One may find it useful to draw a table of values to help one sketch a function. In the case of constant function's it's definitely not necessary, but if you want to it's possible. What you do is draw a table in which the first row will contain various values of x, and the second row will contain the corresponding values of $f(x)$, which will become values of y on the graph.

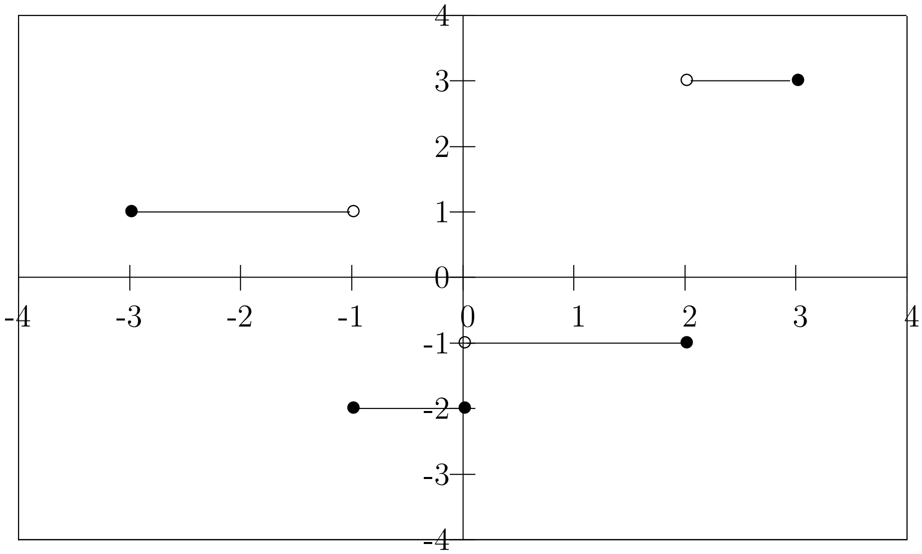
E.g. For this same function ($f(x) = 3$), we can do this simple table:

x	-3	-2	-1	0	1	2	3
$f(x)$	-3	-3	-3	-3	-3	-3	-3

This illustrates that the function always gives -3 as output, regardless of the input. Since constant functions are pretty boring, their tables are also pretty boring, no surprises here.

Step functions (also known as staircase functions)

The next type of functions for us to study is **step functions**. These can best be understood at first through a graph of an example:



The graph consists of several horizontal line segments, with some hollow circles and fill-in circles. Let's say this function is called f. The way to interpret this graph is as follows:

The horizontal lines indicate the function's value. A filled-in circle indicates that point is included. The hollow circle indicates that point isn't included.

So: The left-most line segment is at y value 1, and x values between -3 and -1. So it tells us that the $f(x) = 1$ when x is between -3 and -1. However, it has a filled-in circle at x value -3, and a hollow circle at x value -1. That tells us $f(-3)$ is 1, but $f(-1)$ isn't 1.
We can sum this up as "$f(x) = 1$ if $-3 \leq x < 1$".
The next line segment, at height -2, goes from where x is -1 to where it's 0. And it has filled-in circles at both sides. So it tells us the value of the function is -2 when x is between -1 and 0, including the endpoints:
"$f(x) = -2$ if $-1 \leq x \leq 0$"
Similarly, the other line segments tell us:
"$f(x) = -1$ if $0 < x \leq 2$" and "$f(x) = 3$ if $2 < x \leq 3$".

Things to note:

1. For this to be a function, we can't have more than one filled-circle for the same value of x, because a function must give one output for each input, and if there were more than one filled-circle for the same value of x it would mean more than one value for the function (so it wouldn't be a function).

2. This function's domain is not $\mathbb{R}$, since the function is only defined from -3 to 3. So the domain is $[-3, 3]$.
3. For each value of x between -3 and 3, the function must have a value. So, for example, we couldn't have just hollow circles for some value of x, there has to be one filled-in circle (of course, this is for the end-points, inside the line segments the value of the function is indicated by the line).

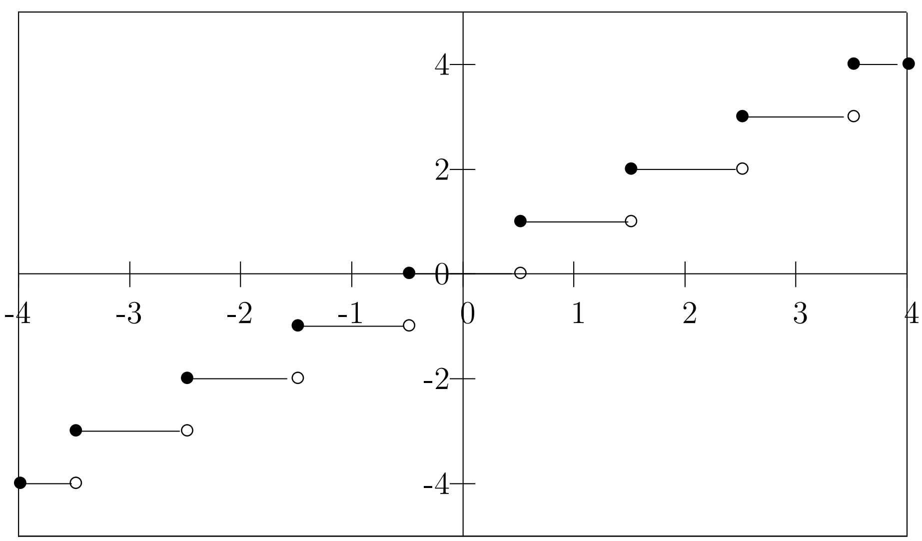
To define the function using letters (as opposed to using a graph), we sum these up using the following notation:
$f : [-3, 3] \to \mathbb{R}$

$$f(x) = \begin{cases} 1 & \text{if } -3 \leq x < 1 \\ -2 & \text{if } -1 \leq x \leq 0 \\ -1 & \text{if } 0 < x \leq 2 \\ 3 & \text{if } 2 < x \leq 3 \end{cases}$$

E.g. Next, here is a function which illustrates why these are also called "staircase functions". Suppose we have a function f with domain $\mathbb{R}$ and codomain $\mathbb{Z}$ (the whole numbers), which maps each real number to the closest whole number (and, as is usual, if it's right in the middle it's rounded up, so 2.5 would get mapped to 3).

$f : \mathbb{R} \to \mathbb{Z}$
$\quad x \mapsto$ the whole number nearest to x

As with the constant functions, our graph will only represent part of the function, since we can't do an infinite graph.



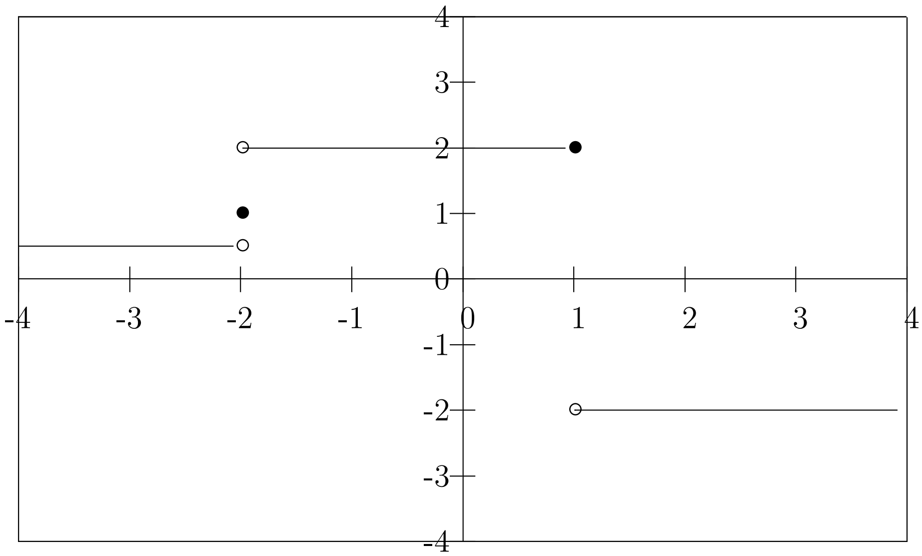
Note: the function was defined as having codomain $\mathbb{Z}$. It could also have been defined with codomain $\mathbb{R}$. Technically, this would make it a different function, but their graphs are the same.

E.g. Draw the graph for the following function:

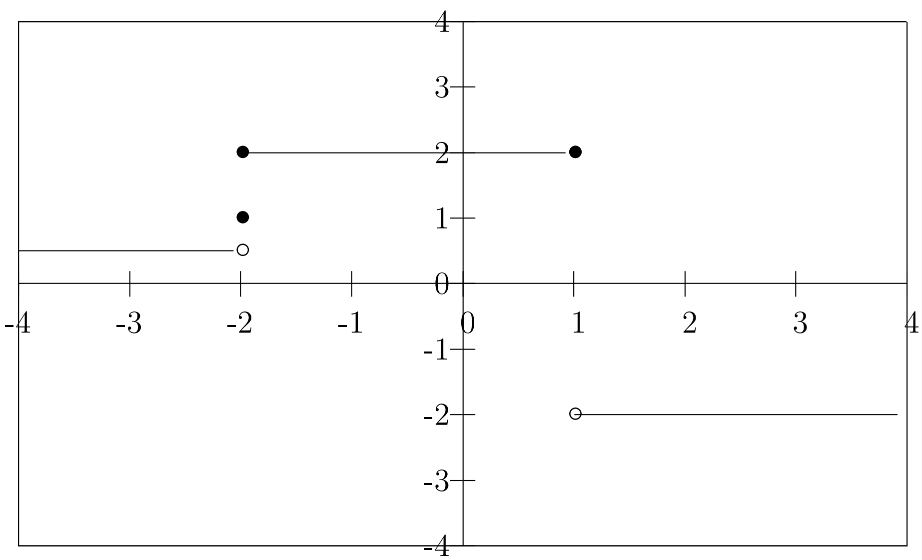
$f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} \frac{1}{2} & \text{if } \ x < -2 \\ 1 & \text{if } \ x = -2 \\ 2 & \text{if } \ -2 < x \leq 1 \\ -2 & \text{if } \ 1 < x \end{cases}$$

Notice that this function's domain is $\mathbb{R}$, so we can't draw the entire graph, we'll just draw the parts which are the most interesting, and we would usually indicate with an arrow that it goes on on either side.



E.g. Here is a very similar graph. However, it's not the same, and the difference is very important.



This isn't actually a function, because for the x value -2 (i.e. when $x = -2$), there are two solid circles, so this would be like a function having two outputs for the same input, which isn't allowed. So it's not a function.

Mapping, images, and preimages

So far we've been using some informal terminology such as inputs and outputs. Here are some formal terms:

The rule which tells us how a function's output depends on its input is called **the function's mapping**. For example, for the constant function of value 2, $f(x) = 2$ is its mapping. Notice we've used the "map" terminology earlier (e.g. "f maps x to 2" in the beginning of the constant functions section, on page 1). As another example, for the step functions the mapping is given according to cases (like at the top of page 5).

If a is an element of the domain of a function f, then $f(a)$ is called **the image of** a **under** f. Usually, the function is known from the context, and it's just called **the image of** a. In other words, the image of a is just the output that we get from the function if we give it a as the input.

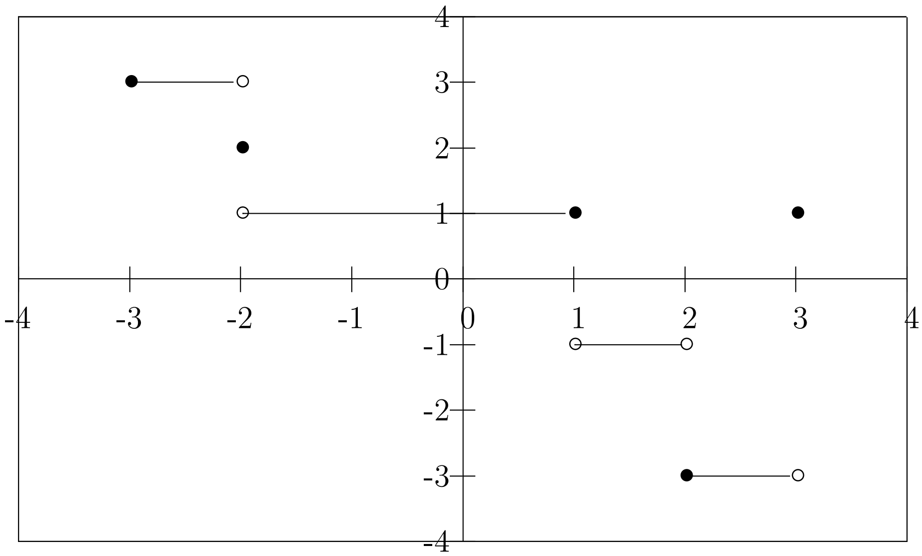
If a is an element of the domain of a function f, and b is its image (i.e. $b = f(a)$), then a is said to be **a preimage of** b **under** f. Again, the function is usually known from context, and we just say a is a preimage of b (or, equivalently, a **is a preimage of** $f(a)$). In other words, a preimage of an element, b, in the codomain is an element in the domain which would give b as output if it was given to the function as an input.

Notes:

1. By the definition of a function (if an element of the domain is given as an input, the function must give exactly one element of the codomain as an output), **every element of the domain has exactly one image**. For example, for the constant function $f(x) = 2$, the image of 1 is 2, since if we put 1 as an input we get 2 as an output. In fact, the image of any number would be 2 with this function (since it always gives 2, regardless of the input).
2. An element of the codomain can have several preimages, one preimage, or no preimages. For example, if we're looking at the constant function $f(x) = 2$, then 0 has no preimages, since no inputs is mapped to 0. On the other hand, 2 has infinitely many images, since all the real numbers (all the elements of the domain) get mapped to 2. We say that the set of preimages of 2 is $\mathbb{R}$.

Practice exercises

#P1. Consider the function, f, with domain $[-3, 3]$ and codomain $\mathbb{R}$ shown in the following graph.



(a) Write down the function's mapping.
(b) Find the image of $\frac{1}{2}$.
(c) Find the image of 2.
(d) Find the image of 3.
(e) Find any preimages of 2.
(f) Find any preimages of 1.

#P2. Consider the following function:
$f : [-4 : 4] \to \mathbb{R}$

$$f(x) = \begin{cases} -2 & \text{if } \; -4 \leq x < 1 \\ \frac{1}{2} & \text{if } \; 1 \leq x < 2 \\ 0 & \text{if } \; x = 2 \\ 1 & \text{if } \; 2 < x \leq 4 \end{cases}$$

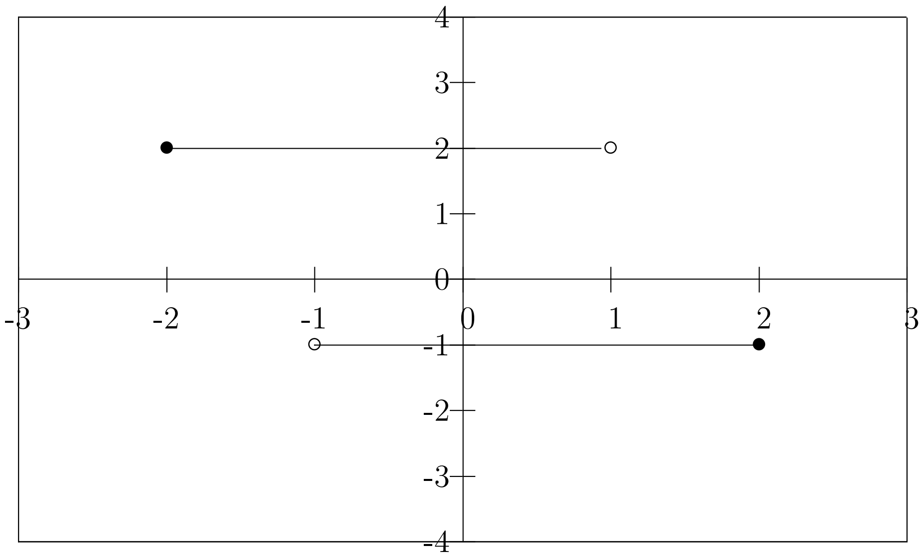
(a) Draw the function's graph.
(b) Find the image of 2.5.
(c) Find the image of 1.
(d) Find any preimages of 0.
(e) Find any preimages of 1.

#3. For each of the following, say whether or not it's a function, and if it's not then explain why.
(a) Given any person's name as input, it gives as output the name of that person's first-born son.
(b) Given any whole number, it gives 1 if the number is even, and 0 if the number is odd.
(c) Given any whole number, it gives the number's divisors.
(d) Given any real number x, it gives $\frac{1}{x}$.
(e) Given any real number, x, it gives $2x$.

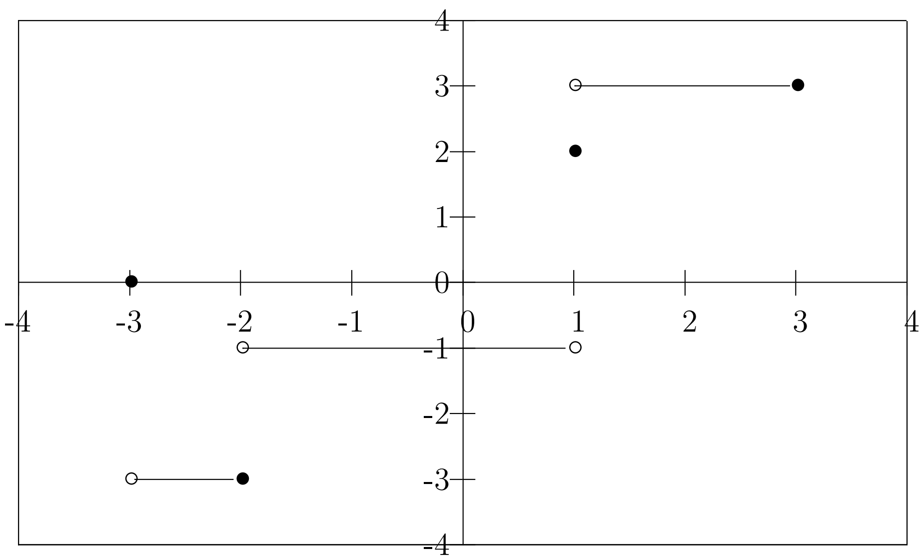
Homework

#1. For each of the following, say whether or not it's a function, and if it's not then explain why.
(a) Given any whole number, it gives $\frac{1}{2}$.
(b) Given any real number, x, it gives x^2.
(c) Given any real number, x, it gives $\sqrt{x}$.
(d) Given any real number, x, it gives the area of a circle with radius x.
(e) Given the name of any person, it gives the first letter of that person's name.
(f) Given any person, it gives the names of that person's parents.
(g) Given any number in the interval $[-1, 1]$, it gives 1 if it's positive, and -1 if it's negative.

#2. Does the following graph represent a function with domain $[-2, 2]$ or not? If not, explain why. If it does, find for the function: (a) the mapping, (b) the image of -2, (c) the image of 1.5, (d) any preimages of 2.

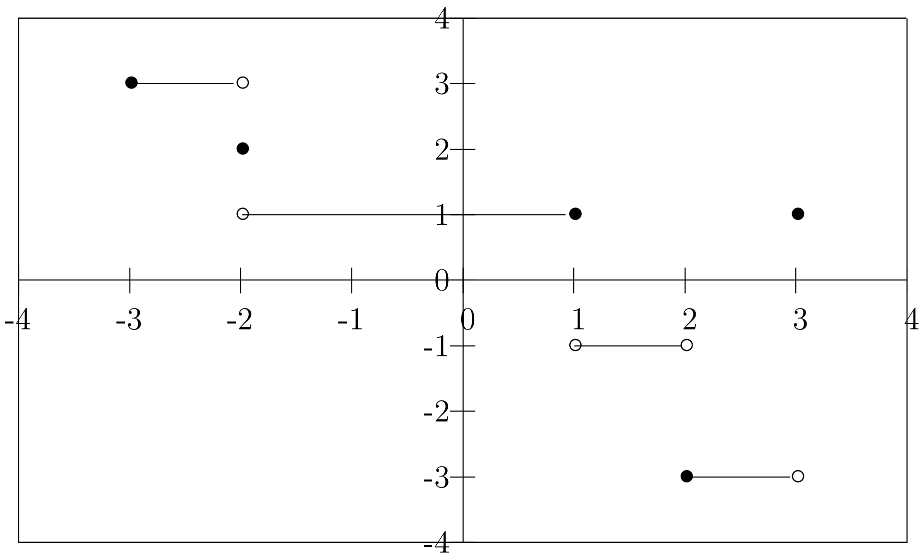


#3. Does the following graph represent a function with domain $[-3, 3]$ or not? If not, explain why. If it does, find for the function: (a) the mapping, (b) the image of -2, (c) the image of 1, (d) any preimages of 0.



Solutions for practice exercises

#P1. Consider the function, f, with domain $[-3,3]$ and codomain $\mathbb{R}$ shown in the following graph.



(a) Write down the function's mapping.

$$f(x) = \begin{cases} 3 & \text{if } -3 \leq x < -2 \\ 2 & \text{if } x = -2 \\ 1 & \text{if } -2 < x \leq 1 \\ -1 & \text{if } 1 < x < 2 \\ -3 & \text{if } 2 \leq x < 3 \\ 1 & \text{if } x = 3 \end{cases}$$

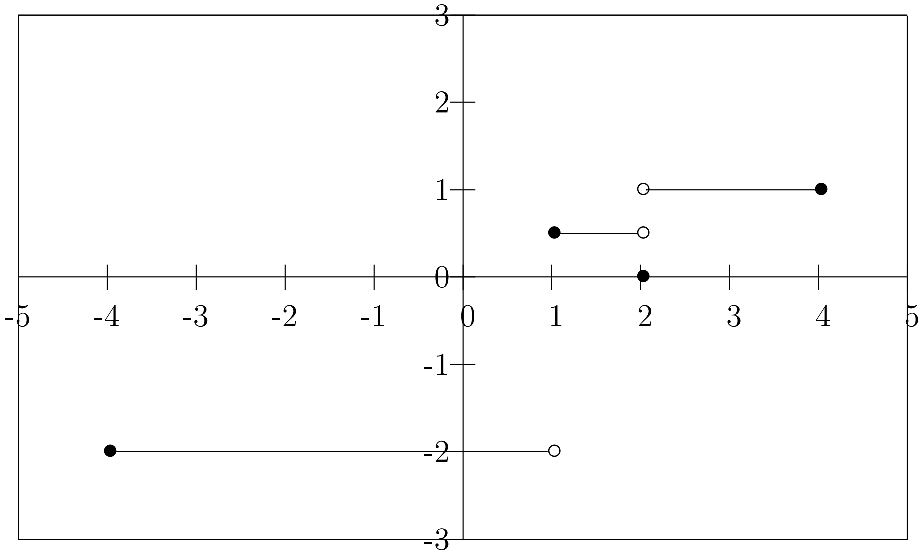
(b) Find the image of $\frac{1}{2}$.
$f(\frac{1}{2}) = 1$
(c) Find the image of 2.
$f(2) = -3$
(d) Find the image of 3.
$f(3) = 1$
(e) Find any preimages of 2.
The only preimage of 2 is -2.
(f) Find any preimages of 1.
The preimages of 1 are all the points in $]-2, 1]$ and 3.

#P2. Consider the following function:

$f : [-4 : 4] \to \mathbb{R}$

$$f(x) = \begin{cases} -2 & \text{if } -4 \le x < 1 \\ \frac{1}{2} & \text{if } 1 \le x < 2 \\ 0 & \text{if } x = 2 \\ 1 & \text{if } 2 < x \le 4 \end{cases}$$

(a) Draw the function's graph.



(b) Find the image of 2.5.

$f(2.5) = 1$

(c) Find the image of 1.

$f(1) = \frac{1}{2}$

(d) Find any preimages of 0.

The only preimage of 0 is 2.

(e) Find any preimages of 1.

The preimages of 1 are all the points in the interval $]1, 4]$.

#3. For each of the following, say whether or not it's a function, and if it's not then explain why.

(a) Given any person's name as input, it gives as output the name of that person's first-born son.

Not a function: some people don't have any sons, so not every input has an output.

(b) Given any whole number, it gives 1 if the number is even, and 0 if the number is odd.

It is a function, since every whole number is either even or odd, but not both.

(c) Given any whole number, it gives the number's divisors.

Not a function, since some whole numbers have more than one divisor, and a function has to have exactly one output for every input.

(d) Given any real number x, it gives $\frac{1}{x}$.

Not a function: since you can't divide by 0, if you input 0 then there is no output, and for it to be a function every input has to have an output.

(e) Given any real number, x, it gives $2x$.

It is a function: for every number x there is exactly one number which equals $2x$.