Logic (part 3)

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Using truth tables

Once we have constructed a truth table for a proposition, we can tell whether it's a tautology, a contradiction or contingent:

A proposition is a tautology if it's true in every row.
A proposition is a contradiction if it's false in every row.
A proposition is contingent if it's true in at least one row and false in at least one row.

We can also formally define logical equivalence:

Two propositions are logically equivalent if they have the same truth value as each other in each row.

E.g. #1. Are $p \vee \neg q$ and $\neg p \Rightarrow \neg q$ logically equivalent?
To save space, we can draw one truth tables for both of them:

p	q	$\neg q$	$p \vee \neg q$	$\neg p$	$\neg p \Rightarrow \neg q$
T	T	F	T	F	T
T	F	T	T	F	T
F	T	F	F	T	F
F	F	T	T	T	T

Now, the two columns we are interested in, in the end, are the ones for our two sentences. And we see that they always have the same truth values as each other, so they are logically equivalent.

Converse, inverse, and contrapositive

Given a conditional proposition (one of the form "If ...then ..."), we can define its converse, inverse, and contrapositive propositions:

Suppose we have a sentence of the form $p \Rightarrow q$. Then:

The inverse of $p \Rightarrow q$ is $\neg p \Rightarrow \neg q$.
The converse of $p \Rightarrow q$ is $q \Rightarrow p$.
The contrapositive of $p \Rightarrow q$ is $\neg q \Rightarrow \neg p$.

E.g. Find the inverse, the converse, and the contrapositive of the proposition "If it's raining then there are clouds in the sky."
Inverse: "If it's not raining, then there aren't clouds in the sky."
Converse: "If there are clouds in the sky, then it's raining."
Contrapositive: "If there aren't clouds in the sky, then it's not raining."

Exercise

#1. Suppose we start with the conditional sentence $p \Rightarrow q$. By drawing and interpreting truth tables, find out:
(a) Whether it and its inverse are logically equivalent.
(b) Whether it and its converse are logically equivalent.
(c) Whether it and its contrapositive are logically equivalent.
(d) Whether its inverse and its converse are logically equivalent.
(e) Whether its inverse and its contrapositive are logically equivalent.
(f) Whether its converse and its contrapositive are logically equivalent.

#2. For each of the following, find the inverse, the converse, and the contrapositive, draw the indicated truth table, and use it to classify the proposition (the one for which you drew the truth table, not the original one) as a tautology, a contradiction, or contingent.

(a) $(p \vee q) \Rightarrow p$, draw the truth table for its converse.
(b) $p \Rightarrow (\neg q \Rightarrow p)$, draw the truth table for its inverse.
(c) $\neg q \Rightarrow p$, draw the truth table for its contrapositive.
(d) $p \Rightarrow (\neg q \vee p)$, draw the truth table for its contrapositive.

#3. Answer each of the following by drawing and interpreting a truth table. If translations are required, use the following key:
p: I like painting.
q: I like quilting.
(a) Is the sentence $(p \vee q) \wedge \neg p$ a tautology?
(b) Are the sentences $\neg(p \wedge q)$ and $\neg p \vee \neg q$ logically equivalent?
(c) Are the sentences "If I don't like quilting, then I like painting" and "If I don't like painting then I like quilting" logically equivalent?
(d) Is the sentence $p \veebar (q \Leftrightarrow \neg p)$ a tautology, a contradiction, or contingent?
(e) Is the contrapositive of $p \Rightarrow (q \Rightarrow p)$ a tautology?

Homework

#H1. For each of the following: (i) translate it into logical notation using the key given, (ii) draw a truth table, (iii) say whether it's a tautology, a contradiction, or contingent, (iv) find the inverse, converse, or contrapositive if so indicated, writing it in logical notation and also translating it into English.

Key:
p: Peter is old.
q: The queen is married.
r: Richard is bossy.

(a) If peter is old, then the queen isn't married. (the converse)
(b) Either the queen is married, or Peter isn't old.
(c) It's not the case both that Peter is old and that Richard is bossy.
(d) If either Peter is old or Richard is bossy, then Peter isn't old. (the contrapositive)

#H2. Draw a truth table for each of the following and use it to say whether the sentence is a tautology, a contradiction, or contingent.
(a) $(p \wedge \neg p) \vee p$
(b) $(p \veebar q) \Rightarrow (p \vee q)$
(c) $(\neg p \vee q) \vee (\neg q \wedge p)$
(d) $p \Leftrightarrow (q \Rightarrow p)$
(e) $(p \wedge q) \wedge (\neg p \Leftrightarrow q)$
(f) $(p \veebar q) \Rightarrow \neg(p \wedge q)$
(g) $(p \veebar q) \Leftrightarrow (p \Leftrightarrow q)$
(h) $\neg\neg p \Rightarrow p$

#H3. Find out, by means of truth tables, whether the following sentences are equivalent to each other (i.e. do this for each pair of sentences).

(a) $\neg p \wedge q$ and $\neg(p \vee \neg q)$.
(b) $\neg p \vee q$ and $\neg(p \vee \neg q)$.
(c) $p \Rightarrow \neg q$ and $\neg(p \wedge q)$.
(d) $p \Rightarrow (\neg p \Rightarrow p)$ and $p \vee \neg p$.
(e) $\neg p \veebar q$ and $p \Rightarrow \neg q$.
(f) $\neg(p \veebar \neg q)$ and $\neg p \Leftrightarrow \neg q$.
(g) $p \Leftrightarrow (p \wedge q)$ and $\neg q \Rightarrow q$.

<u>Solutions for in-class exercises</u>

#1. A sentence is logically equivalent to its contrapositive (c). And its inverse is logically equivalent to its converse (d). The others are not.

#2. For each of the following, find the inverse, the converse, and the contrapositive, draw the indicated truth table, and use it to classify the proposition (the one for which you drew the truth table, not the original one) as a tautology, a contradiction, or contingent.

(a) $(p \vee q) \Rightarrow p$, draw the truth table for its converse.
Inverse: $\neg(p \vee q) \Rightarrow \neg p$
Converse: $p \Rightarrow (p \vee q)$
Contrapositive: $\neg p \Rightarrow \neg(p \vee q)$

p	q	$p \vee q$	$p \Rightarrow (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

The converse is a tautology.

(b) $p \Rightarrow (\neg q \Rightarrow p)$, draw the truth table for its inverse.
Inverse: $\neg p \Rightarrow \neg(\neg q \Rightarrow p)$
Converse: $(\neg q \Rightarrow p) \Rightarrow p$
Contrapositive: $\neg(\neg q \Rightarrow p) \Rightarrow \neg p$

p	q	$\neg q$	$\neg q \Rightarrow p$	$\neg(\neg q \Rightarrow p)$	$\neg p$	$\neg p \Rightarrow \neg(\neg q \Rightarrow p)$
T	T	F	T	F	F	T
T	F	T	T	F	F	T
F	T	F	T	F	T	F
F	F	T	F	T	T	T

The inverse is contingent.

(c) $\neg q \Rightarrow p$, draw the truth table for its contrapositive.
Inverse: $\neg\neg q \Rightarrow \neg p$ (equivalently, $q \Rightarrow \neg p$, though the way I have it is more literal)
Converse: $p \Rightarrow \neg q$
Contrapositive: $\neg p \Rightarrow \neg\neg q$

p	q	$\neg q$	$\neg\neg q$	$\neg p$	$\neg p \Rightarrow \neg\neg q$
T	T	F	T	F	T
T	F	T	F	F	T
F	T	F	T	T	T
F	F	T	F	T	F

The contrapositive is contingent.

(d) $p \Rightarrow (\neg q \vee p)$, draw the truth table for its contrapositive.
Inverse: $\neg p \Rightarrow \neg(\neg q \vee p)$
Converse: $(\neg q \vee p) \Rightarrow p$
Contrapositive: $\neg(\neg q \vee p) \Rightarrow \neg p$

p	q	$\neg q$	$\neg q \vee p$	$\neg(\neg q \vee p)$	$\neg p$	$\neg(\neg q \vee p) \Rightarrow \neg p$
T	T	F	T	F	F	T
T	F	T	T	F	F	T
F	T	F	F	T	T	T
F	F	T	T	F	T	T

The contrapositive is a tautology.

#3. Answer each of the following by drawing and interpreting a truth table. If translations are required, use the following key:
p: I like painting.
q: I like quilting.

(a) Is the sentence $(p \vee q) \wedge \neg p$ a tautology?

p	q	$(p \vee q)$	$\neg p$	$(p \vee q) \wedge \neg p$
T	T	T	F	F
T	F	T	F	F
F	T	T	T	T
F	F	F	T	F

No, it's not a tautology (it's contingent).

(b) Are the sentences $\neg(p \wedge q)$ and $\neg p \vee \neg q$ logically equivalent?

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

They are logically equivalent.

(c) Are the sentences "If I don't like quilting, then I like painting" and "If I don't like painting then I like quilting" logically equivalent?
These are $\neg q \Rightarrow p$ and $\neg p \Rightarrow q$

p	q	$\neg q$	$\neg q \Rightarrow p$	$\neg p$	$\neg p \Rightarrow q$
T	T	F	T	F	T
T	F	T	T	F	T
F	T	F	T	T	T
F	F	T	F	T	F

They are logically equivalent.

(d) Is the sentence $p \veebar (q \Leftrightarrow \neg p)$ a tautology, a contradiction, or contingent?

p	q	$\neg p$	$q \Leftrightarrow \neg p$	$p \veebar (q \Leftrightarrow \neg p)$
T	T	F	F	T
T	F	F	T	F
F	T	T	T	T
F	F	T	F	F

The sentence is contingent.

(e) Is the contrapositive of $p \Rightarrow (q \Rightarrow p)$ a tautology?
The contrapositive is $\neg(q \Rightarrow p) \Rightarrow \neg p$

p	q	$q \Rightarrow p$	$\neg(q \Rightarrow p)$	$\neg p$	$\neg(q \Rightarrow p) \Rightarrow \neg p$
T	T	T	F	F	T
T	F	T	F	F	T
F	T	F	T	T	T
F	F	T	F	T	T

The contrapositive is a tautology.