Distances, midpoints, parallel and perpendicular lines
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The distance between two points

To find the distance between two points, we use Pythagoras' theorem.
E.g. Find the distance between $(2, 5)$ and $(-1, 3)$.
The vertical distance between the two points is $5 - 3 = 2$.
The horizontal distance between the two points is $2 - (-1) = 3$.
So the diagonal distance between them is $\sqrt{2^2 + 3^2} = \sqrt{13}$.
Note that, unlike when calculating the gradient, the signs of the distances don't matter because we square them anyway.

This can be written down as a formula:
The distance between (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Midpoint between two lines

The midpoint between two points is their midpoint both horizontally and vertically.

E.g. Find the midpoint of $(2, 5)$ and $(-1, 3)$.
The x coordinate of the midpoint will be the midpoint between 2 and -1. This is their average: $\frac{2-1}{2} = \frac{1}{2}$.
Likewise, the y coordinate will be the midpoint between 5 and 3, which is 4. So the midpoint will be $(\frac{1}{2}, 4)$.

The midpoint of (x_1, y_1) and (x_2, y_2) is $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$.

Parallel lines

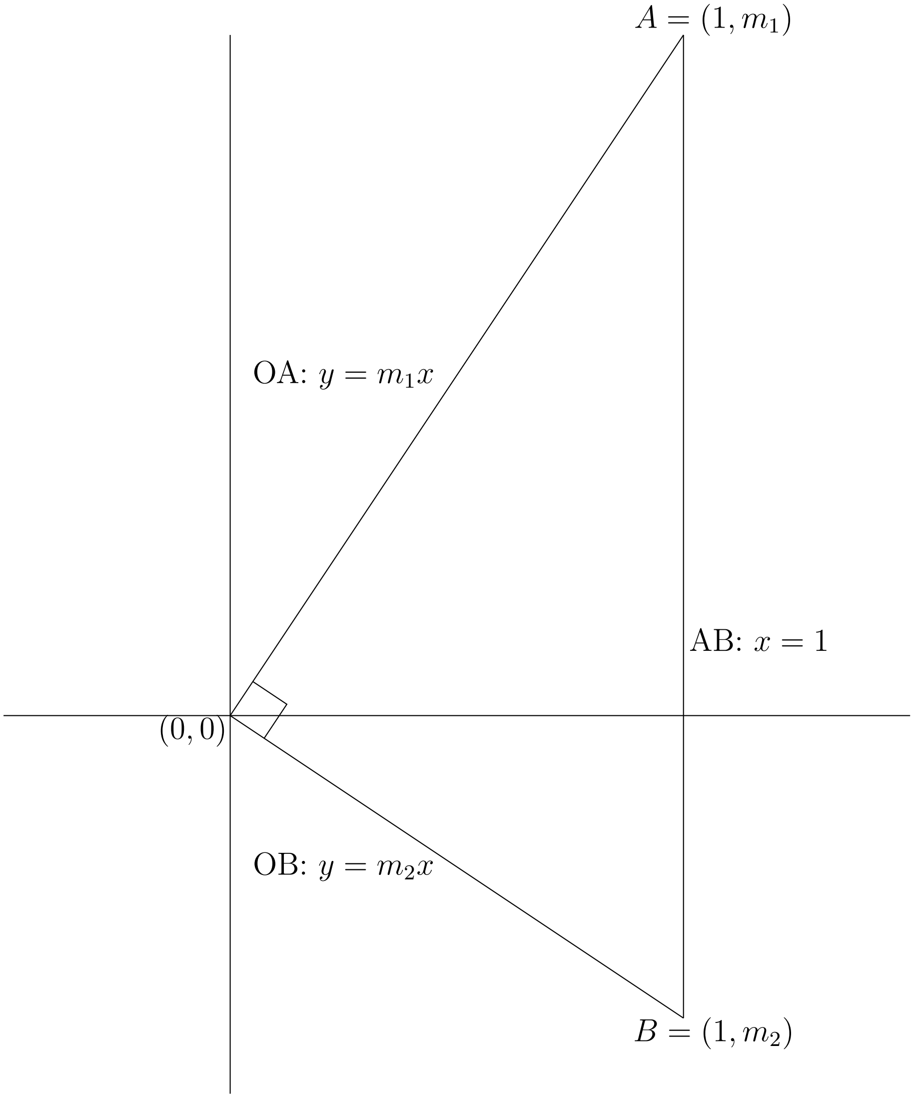
As we've seen before, two lines are parallel if they have the same gradient. We can use this to find lines parallel to a given line.

E.g. Find the equation of the line which is parallel to $y = 2x+1$ and which passes through $(-1, -5)$.
Its gradient will be 2, since that's the gradient of $y = 2x + 1$. So it will be $y = 2x + b$. To find b, we substitute our point:
$-5 = 2(-1) + b$
$-3 = b$
So its equation is $y = 2x - 3$.

Perpendicular lines

There is also a relationship between the gradients of two lines which are perpendicular to each other. To find out what it is, we'll use Pythagoras' theorem.

Since we're only interested in finding the relationship between their gradients, we can consider lines which pass through the origin. So consider two lines which are perpendicular to each other:
l_1 has equation $y = m_1 x$.
l_2 has equation $y = m_2 x$.



The two lines intersect at the origin, $(0, 0)$ (let's call it O, as usual).
The point $(1, m_1)$ (which we'll call A) lies on l_1, and the point $(1, m_2)$ (which we'll call b) lies on l_2.
Let's consider the triangle formed by these three points.
Since AOB is a right angle, Pythagoras' theorem tells us:
$OA^2 + OB^2 = AB^2$.
We can calculate each of these distances and substitute them into this equation:
$OA = \sqrt{1^2 + m_1^2} = \sqrt{1 + m_1^2}$
Therefore, $OA^2 = 1 + m_1^2$
$OB = \sqrt{1^2 + m_2^2} = \sqrt{1 + m_2^2}$
Therefore, $OB^2 = 1 + m_2^2$
$AB = \sqrt{(1-1)^2 + (m_2 - m_1)^2} = \sqrt{(m_2 - m_1)^2}$

Therefore, $AB^2 = (m_2 - m_1)^2$
So, using Pythagoras' theorem, we get:
$(1 + m_1^2) + (1 + m_2^2) = (m_2 - m_1)^2$
$1 + m_1^2 + 1 + m_2^2 = m_2^1 - 2m_1m_2 + m_1^2$
$2 = -2m_1m_2$
$-1 = m_1m_2$

So, if two lines are perpendicular, their gradients multiplied together give -1.

This can also be written:
$m_2 = -\frac{1}{m_1}$

We can use this to find equations of lines perpendicular to a given line.

E.g. Find the equation of the line which is perpendicular to $x + 2y = 3$ and which passes through $(-1, -2)$.
$x + 2y = 3$ can be written as $y = -\frac{1}{2}x + \frac{3}{2}$, so it has gradient $-\frac{1}{2}$.
Therefore, the line perpendicular to it will have gradient $m = -\dfrac{1}{-\frac{1}{2}} = 2$. So it will have equation $y = 2x + b$.
To find b, we subsitute out point into the equation:
$-2 = 2(-1) + b$
$-4 = b$
Therefore, its equation is $y = 2x - 4$.

<u>Exercises</u>

#1. Suppose the line l_1 passes through the points $(-1, 5)$ and $(-5, -3)$.
(a) Find the equation for l_1, and express it in both forms.
(b) Find l_1's intersections with the axes.
(c) Suppose l_2 is perpendicular to l_1 and passes through the point $(5, -3)$. Find the equation of l_2.
(d) Find the point of intersection of l_1 and l_2.
(e) Draw a diagram representing the lines and intersections found above.
(f) Find the distance between the origin (the point $(0, 0)$) and the point of intersection you found in (d).

#2. Suppose f is a linear function whose graph passes through the points $(1, -\frac{3}{2})$ and $(-2, 2)$.
(a) Find f's mapping.
(b) Find the inverse function of f. Check your answer by finding $ff^{-1}(x)$.
(c) Find the image of 5 under f.
(d) Find the preimage of $\frac{7}{2}$ under f.

#3. Find the equation of a line which is perpendicular to $y = 3$ and which passes through the point $(-2, 4)$.

#4. Find the equation of a line parallel to $x + 3y = 9$ and which passes through the point $(4, 0)$.

#5. Find the midpoint between the point $(2, -3)$ and the point where the line $x - 2y = 8$ intersects the x axis.

Solutions

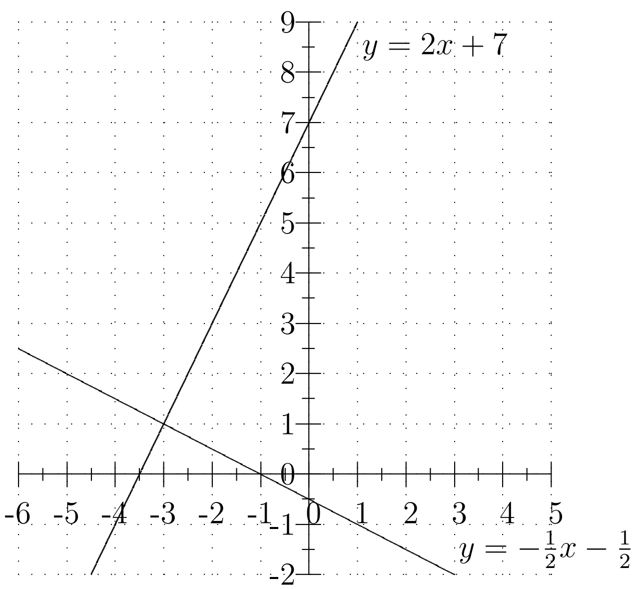
#1. Suppose the line l_1 passes through the points $(-1, 5)$ and $(-5, -3)$.
(a) Find the equation for l_1, and express it in both forms.
$m = \dfrac{-3-5}{-5-(-1)} = \dfrac{-8}{-4} = 2$
So $y = 2x + b$. Substituting $(-1, 5)$, we get:
$5 = 2(-1) + b$
$7 = b$
So l_1 has equation $y = 2x + 7$, which can also be written $-2x + y = 7$.

(b) Find l_1's intersections with the axes.
The intersection with the y axis is at $y = 7$.
The intersection with the x axis is at $x = -\frac{7}{2}$.

(c) Suppose l_2 is perpendicular to l_1 and passes through the point $(5, -3)$. Find the equation of l_2.
The gradient of l_2 will be $-\frac{1}{2}$. So $y = -\frac{1}{2}x + b$ Substituting $(5, -3)$, we get
$-3 = -\frac{1}{2}(5) + b$
$-\frac{1}{2} = b$
So l_2 has equation $y = -\frac{1}{2}x - \frac{1}{2}$.

(d) Find the point of intersection of l_1 and l_2.
Since we already have both in the form $y = mx + b$, we can just set them equal to each other:
$2x + 7 = -\frac{1}{2}x - \frac{1}{2}$
$\frac{5}{2}x = -\frac{15}{2}$
$x = -3$
Substituting this back in, we get $y = 2(-3) + 7 = 1$.
So the point of intersection is $(-3, 1)$.

(e) Draw a diagram representing the lines and intersections found above.



(f) Find the distance between the origin (the point $(0, 0)$) and the point of intersection you found in (d).
The distance is $\sqrt{3^2 + 1^2} = \sqrt{10}$

#2. Suppose f is a linear function whose graph passes through the points $(1, -\frac{3}{2})$ and $(-2, 2)$.
(a) Find f's mapping.
We just need to find $f(x)$, which means finding the equation of the line (since a function's graph is $y = f(x)$).
$m = \dfrac{2-(-\frac{3}{2})}{-2-1} = \dfrac{\frac{7}{2}}{-3} = -\dfrac{7}{6}$
So $f(x) = -\frac{7}{6}x + b$
Substituting $(-2, 2)$, we get:
$2 = -\frac{7}{6}(-2) + b$
$-\frac{1}{3} = b$
So $f(x) = -\frac{7}{6}x - \frac{1}{3}$.

(b) Find the inverse function of f. Check your answer by finding $ff^{-1}(x)$.
To find the inverse funciton, we set $y = f(x)$ and solve for x (i.e. find the preimage of y):
$y = -\frac{7}{6}x - \frac{1}{3}$
$6y = -7x - 2$
$7x = -6y - 2$
$x = -\frac{6}{7}y - \frac{2}{7}$
So $f^{-1}(y) = -\frac{6}{7}y - \frac{2}{7}$.
This can also be written:
$f^{-1}(x) = -\frac{6}{7}x - \frac{2}{7}$.
To check we can calculate $ff^{-1}(x)$ and see if it gives x:
$ff^{-1}(x) = f(-\frac{6}{7}x - \frac{2}{7})$
$= -\frac{7}{6}(-\frac{6}{7}x - \frac{2}{7}) - \frac{1}{3}$
$= \frac{7}{6}\frac{6}{7}x + \frac{7}{6}\frac{2}{7} - \frac{1}{3}$
$= x + \frac{1}{3} - \frac{1}{3}$
$= x$
So our answer was right.

(c) Find the image of 5 under f.
$f(5) = -\frac{7}{6}(5) - \frac{1}{3} = -\frac{37}{6}$

(d) Find the preimage of $\frac{7}{2}$ under f.
Finding preimages is easy once you have the inverse function:
Preimage of $\frac{7}{2} = f^{-1}(\frac{7}{2}) = -\frac{6}{7}(\frac{7}{2}) - \frac{2}{7} = -\frac{23}{7}$.

#3. Find the equation of a line which is perpendicular to $y = 3$ and which passes through the point $(-2, 4)$.
$y = 3$ is horizontal, so a line perpendicular to it would be horizontal. Since it passes through $(-2, 4)$, it will be $x = -2$.

#4. Find the equation of a line parallel to $x + 3y = 9$ and which passes through the point $(4, 0)$.
Since it's parallel, it will have equation $x + 3y = c$. Since it passes through $(4, 0)$, we have:
$4 + 3(0) = c$
$7 = c$
Therefore, its equation is $x + 3y = 7$.

#5. Find the midpoint between the point $(2, -3)$ and the point where the line $x - 2y = 8$ intersects the x axis.
$x - 2y = 8$ intersects the x axis at $x = 8$, so the point is $(8, 0)$.
The midpoint, then, is $(\frac{2+8}{2}, \frac{-3+0}{2}) = (5, -\frac{3}{2})$.