Polynomial inequalities

Mr. Neeman. 10A, August 23, 2011

Polynomial inequalities are like quadratic inequalities, except instead of a quadratic expression we have a higher order polynomial. The strategy we use will be similar. Recall. from when we did polynomial equations, that every polynomial can be factorized into a product of linear terms (terms of the form $ax + b$, where a and b are real numbers) and irreducible quadratics (quadratics which can't be factorized). For example: $x^4 - x^3 - 2x - 4 = (x+1)(x-2)(x^2+2)$, a product of two linear terms and one irreducible quadratic terms. So, when faced with a polynomial inequality, we proceed to:

Step 1. Move all the terms to one side.
Step 2. Factorize the polynomial as far as possible, so it's a product of linear terms and irreducible quadratic terms.

E.g. (first two steps) $x^4 \leq x^3 + 2x + 4$
First, we move all the terms to the left: $x^4 - x^3 - 2x - 4 \leq= 0$
Next, we factorize the left hand side $(x+1)(x-2)(x^2+2) \leq 0$ (we do it as when we did polynomial equations).
Now, we consider the signs of each of the three factors. $x^2 + 2$ is irreducible, meaning it has no roots. So we know it's either always positive or always negative. We can try some value, for example $x = 0$ to see that it's positive, so $x^2 + 2$ is always positive. Therefore, the sign of our overall product will be the same as the sign of $(x+1)(x-2)$. (If we had had a quadratic which was always negative, if we divided by it we'd have to flip the inequality sign. In such cases, one can also just factor out a -1 and divide by that, but of course you have to flip the inequality sign since -1 is negative.) A good way to think about this is that, since $x^2 + 2 \neq 0$, whatever x is, we can divide by $x^2 + 2$, and since it's positive the inequality sign stays the same. So we get:
$(x+1)(x-2) \leq 0$
This is just a quadratic inequality, which we know how to do. The solution is $-1 \leq x \leq 2$.

One complication which can arise is when we have repeated roots. Since we already did factorization before, and what's new is the inequality sign, I'll consider a polynomial which is already factorized.

E.g. $(x-1)^2(x+2) \geq 0$
An easy way to do this is using a table. The key values are the roots: $x = 1$ and $x = -2$

	$x < -2$	$x = -2$	$-2 < x < 1$	$x = 1$	$x > 1$
$(x-1)$	negative	negative	negative	0	positive
$(x-1)^2$	positive	positive	positive	0	positive
$(x+2)$	negative	0	postive	positive	positive
$(x-1)^2(x+2)$	negative	0	positive	0	positive

So we see our inequality is satisfied for all the columns except the first one. So our answer is $x \geq -2$.

If, one the other hand, our inequality had been $(x-1)^2(x+2) > 0$, then it would only be the 3rd and the 5th columns. So the inequality would be satisfied when $-2 < x < 1$ and when $x > 1$.

E.g. $x(x-1)(x+2)^3(2x^2+3x+20) > 0$
This one looks rather complicated. The quadratic term is irreducible. This can be confirmed by finding the discriminant or by completing the square. Since it's irreducible, and positive for some values (e.g. with $x = 0$, it's 20), it's always positive, so we can divide both sides by $(2x^2 + 3x + 20)$, getting:
$x(x-1)(x+2)^3 > 0$
One way to solve this is to do the table, with the key values being the roots: $x = -2$, $x = 0$, and $x = 1$. The table would have the following columns: $x < -2$, $x = -2$, $-2 < x < 0$, $x = 0$, $0 < x < 1$, $x = 1$, and $x > 1$. However, we can save ourselves some work by noticing that at the roots, the polynomial is zero, whereas we're looking for it to be strictly greater than zero, so the roots won't be part of our solution. So we only need to look at $x < -2$, $-2 < x < 0$, $0 < x < 1$, and $x > 1$. We find that these give us respectively, negative, positive, negative, and positive. So the inequality is satisfied for $-2 < x < 0$ and for $x > 1$.

E.g. $x(x-1)(x+2)^3(2x^2+3x+20) \geq 0$
This is the same as the previous one, except now it's greater than or equal, instead of strictly greater than. So the solution will be same as before, except with all the roots as well, since that's where the polynomial becomes zero. So $-2 \leq x \leq 0$ or $x \geq 1$.

E.g. $-(x^2+4)(x^2+2x+10) \geq 0$
Here, we can first divide both sides by -1, which gives us
E.g. $(x^2+4)(x^2+2x+10) \leq 0$
Now, both of these quadratics are irreducible and positive, so they're each always positive, and their product is therefore also always positive. Therefore, the inequality has no solutions.

In these exercises I give the polynomials already factored, so you can focus on the inequality aspects rather than factorization. However, you need to be able to do the factorization for the quizzes and exams.
<u>Practice exercises</u>

#P1. $-x(x+1)(x^2+3) \leq 0$

#P2. $(x+1)(x-1)(x+3) > 0$

#P3. $x(x+2)^2(x-1) > 0$

<u>Homework</u>

#H1. $x^2(x+3)(2x^2+4x+20) \leq 0$

#H2. $(x-1)^3(x+4)(x+1)^2 > 0$

#H3. $x(x^2-4x+8) \geq 0$

#H4. $(x+2)(x-3)^2 \leq 0$

Solutions for practice problems

#P1. $-x(x+1)(x^2+3) \leq 0$
The quadratic term is always positive, so we can divide both sides by it, and we get:
$-x(x+1) \leq 0$
This means $0 \leq x(x+1)$.
So we're looking for when both are positive, or when both are negative. This means $x \geq 0$ or $x \leq -1$. Notice that the roots are included, because our inequality isn't strict.

#P2. $(x+1)(x-1)(x+3) > 0$
Here the inequality is strict, so the roots are excluded. Now, the roots are -3, -1, and 1. So there are 4 regions: $x < -3$, $-3 < x < -1$, $-1 < x < 1$, and $x > 1$. We can calculate the sign of the polynomial for each region, using a table or a different method, and we find that it's positive when $-3 < x < -1$ and when $x > 1$, so this is our answer.

#P3. $x(x+2)^2(x-1) > 0$
Since this is strict inequality, the roots will be excluded. The roots are -2, 0, and 1. Now, this also gives us the following 4 intervals: $x < -2$, $-2 < x < 0$, $0 < x < 1$, and $x > 1$. For each one we can find the sign of the polynomial, and we find it's positive for all of them except $0 < x < 1$. So our answer will be $x < -2$, $-2 < x < 0$ or $x > 1$.