Finding equations of lines, and an alternative form for equations of lines

Mr. Neeman, 10A. November 3, 2011

Recall that a straight line has an equation of the form $y = mx + b$, where m and b are constants. m is the line's gradient, and b is where it intersects the y-axis, since $y = b$ when $x = 0$.

As we know, we can draw a straight line through any two points. This means that, given two points, we can find the equation of the line. Likewise, if we know one point and the line's gradient, we can find its equation.

Finding a line's equation, given its gradient and one point

If we know the line's gradient, all we need to do is find b. This can be done by substituting our points coordinates for x and y in the line's equation, and then solving for b. This works because a point (x, y) is on a given line if and only if x and y satisfy the line's equation.

E.g. Find the equation of the line with gradient 2 which passes through the point $(-1, 2)$.
Since the gradiente, m, is 2, the equation will be:
$y = 2x + b$, and we need to find b.
So we take our point $(-1, 2)$ and substitute it into the equation: $x = -1$ and $y = 2$.
$2 = 2(-1) + b$
$2 = -2 + b$
$4 = b$
Therefore, the equation is $y = 2x + 4$.

What we've just done can also be encoded in a formula. Suppose we have a line with gradient m and passing through the point (x_1, y_1). Then the line's equation will be $y - y_1 = m(x - x_1)$. However, it's preferable to just remember the logic of how to find the line's equation rather than memorize another formula.

Finding a line's equation, given two points

If we're, instead, given two points the lines goes through, we first find the gradient m, and then use the gradient and one of the two points to find the equation of the line, using the procedure above.
Recall:
$\text{gradient} = m = \dfrac{\text{change in } y}{\text{change in } x} = \dfrac{y_2 - y_1}{x_2 - x_1}$
E.g. Find the equation of the line passing through $(2, 7)$ and $(5, -1)$.
First, we find m:
$m = \dfrac{-1 - 7}{5 - 2} = \dfrac{-8}{3}$
Now, we can choose one of the two points and substitute it into the equation in order to find b:
$y = mx + b$
$y = -\frac{8}{3}x + b$, since $m = -\frac{8}{3}$
$7 = -\frac{8}{3}2 + b$, since $(2, 7)$ is on the line.
$7 + \frac{16}{3} = b$
$\frac{37}{3} = b$
Therefore, the equation of the line is $y = -\frac{8}{3}x + \frac{37}{3}$
We check our answer by verifying that our second point, $(5, -1)$, also satisfied this equation:
$-1 = -\frac{8}{3}5 + \frac{37}{3}$
$-1 = \frac{-40+37}{3}$
$-1 = \frac{-3}{3}$, which we know is true. So the equation is satisfied, as it should be.

There is also an formula for this, but it's even more complicated. It's obtained by substituting the formula for the gradient into the formula we had before:
$y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$
If we have a the line in a diagram, we can just pick any two points on the line and use them to find the equation of the line as above. If one of the points you choose is on the y axis, that will tell you directly the value of b.

An alternative form for equations of lines

You may, by now, have noticed that $y = mx + b$ includes all sorts of diagonal lines (if $m \neq 0$), and also horizontal lines (if $m = 0$). However, it doesn't include any vertical lines, since those would have an infinite gradient.

Still, vertical lines are important, and they do have equations. We can see what they are through an analogy with horizontal lines:

Consider the horizontal line passing through the points $(1, 3)$ and $(4, 3)$. Its gradient is zero. This can be found using the formula for gradient, or simply by the intuitive meaning of gradient. So the line will have an equation $y = b$, and since our points have y coordinate 3, we know $b = 3$. So its equation is $y = 3$. In other words, the line consists of all point whose y coordinate is 3.

Likewise, the vertical like passing through $(-2, 4)$ and $(-2, 8)$ will consist of all the points whose x coordinate is -2. So its equation will simply be $x = -2$. If you were to try to calculate the gradient, you would get a division by zero, so that's no help. Instead you have to know that all the points on a vertical line have the same x value.

$x = c$ is a vertical line, of all points with x coordinate c.
$y = c$ is a horizontal line, of all points with y coordinate c.

Though writing the equation of straight lines as $y = mx + b$ is very useful, it doesn't include the vertical lines. So there is an alternative format which is often used, and which includes all straight lines:

General form for the equation of a straight line:
$ax + by = c$, where a, b, and c are constants, and a and b aren't both zero

Note, of course, that the b here doesn't have the same meaning as the b in $y = mx + b$.

This form has the advantage that it's very easy to find the intersections with the axes:
If $x = 0$, then $by = c$, so $y = \frac{c}{b}$ (unless $b = 0$, of course).
If $y = 0$, then $ax = c$, so $x = \frac{c}{a}$ (unless $a = 0$, of course).

To convert between one form and the other we just move terms from one side to the other.

E.g. Express $y = \frac{2}{3}x - 2$ in the form $ax + by = c$.
$-\frac{2}{3}x + y = -2$
$-2x + 3y = -6$
Note that there are many equivalent ways (multiplying the whole equation by a constant), so it can also be written:
$2x - 3y = 6$
It can even be left as $-\frac{2}{3}x + y = -2$ if you don't mind fractions.

E.g. Express $2x + 3y = 4$ in the form $y = mx + b$.
We simply solve for y
$3y = -2x + 4$
$y = -\frac{2}{3}x + \frac{4}{3}$

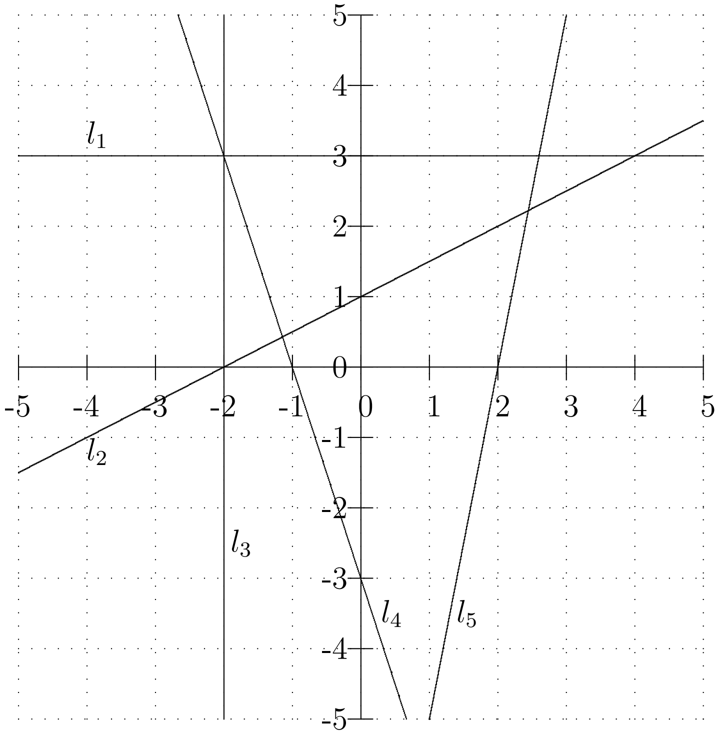
Practice problems

#1. For each of the following, find the line's equation, express it in both forms ($y = mx + c$ and $ax + by = c$) if possible, find the intersections with the axes, and sketch it in a diagram.

(a) L_a: a line with gradient -1, passing through $(2, -3)$.
(b) L_b: a line with gradient $\frac{3}{2}$, passing through $(1, 2)$.
(c) L_c: a line passing through $(4, 1)$ and $(0, 0)$
(d) L_d: a line passing through $(-2, -2)$ and $(1, -2)$
(e) L_e: a line passing through $(-1, 2)$ and $(-1, -3)$.

#2. Consider the line with equation $y = -3x - 1$.
(a) Find its intersections with the axes.
(b) Find a point which lies on this line and isn't on one of the axes.
(c) Does the point $(-2, -7)$ lie on this line?

#3. Find the equations of the lines shown in the following diagram. Express each one in both forms, if possible.



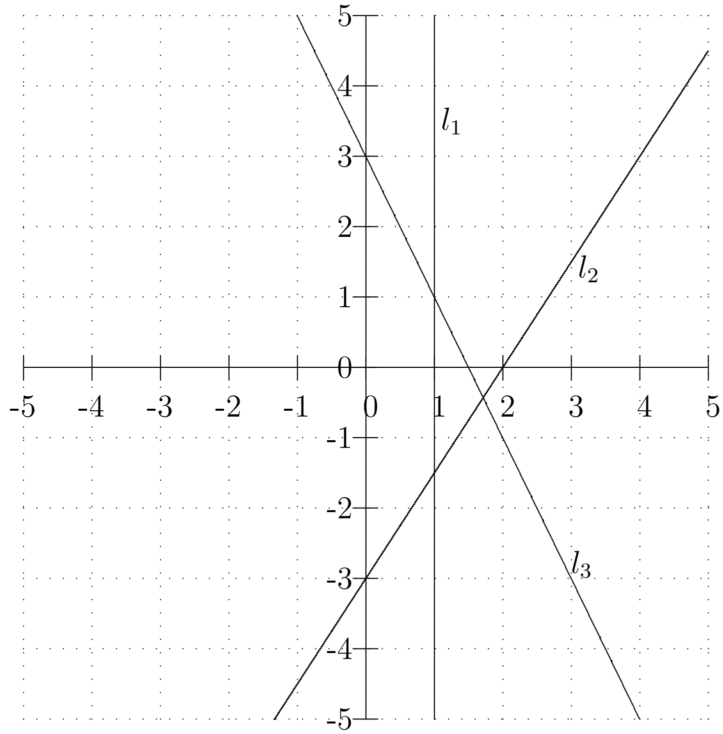
Homework

#1. For each of the following, find the line's equation, express it in both forms ($y = mx + c$ and $ax + by = c$) if possible, find the intersections with the axes, and sketch it in a diagram.

(a) L_a: a line with gradient 2, passing through $(1, -2)$.
(b) L_b: a line with gradient $-\frac{1}{3}$, passing through $(6, 3)$.
(c) L_c: a line passing through $(3, 0)$ and $(-1, -6)$
(d) L_d: a line passing through $(3, -1)$ and $(3, 3)$

#2. Consider the line with equation $y = 2x - 5$. For each of the following points, check whether it lies on the line or not.
(a) $(1, 2)$
(b) $(-2, -9)$
(c) $(3, -1)$

#3. Find the equations of the lines shown in the following diagram. Express each one in both forms, if possible.



Solutions for practice problems

#1. For each of the following, find the line's equation, express it in both forms ($y = mx + c$ and $ax + by = c$) if possible, find the intersections with the axes, and sketch it in a diagram.

(a) L_a: a line with gradient -1, passing through $(2, -3)$.
$-3 = -2 + b$
$-1 = b$
So it's: $y = -x - 1$, which can also be written $x + y = -1$

(b) L_b: a line with gradient $\frac{3}{2}$, passing through $(1, 2)$.
$2 = \frac{3}{2} + b$
$\frac{1}{2} = b$
So it's $y = \frac{3}{2}x + \frac{1}{2}$, which can also be written $-3x + 2y = 1$

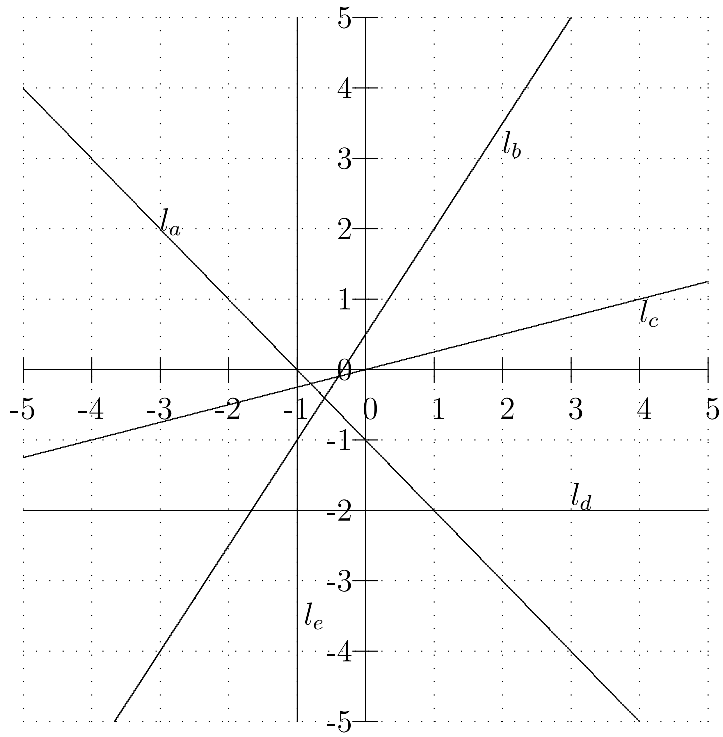
(c) L_c: a line passing through $(4, 1)$ and $(0, 0)$

$$m = \frac{0 - 1}{0 - 4} = \frac{1}{4}$$

Since it passes through $(0, 0)$, $b = 0$.
So it's $y = \frac{1}{4}x$, which can also be written $x - 4y = 0$

(d) L_d: a line passing through $(-2,-2)$ and $(1,-2)$
$m = \dfrac{-2-(-2)}{1-(-2)} = 0$
$-2 = b$
So it's $y = -2$ (we could have found this by inspection), which already in the form $ax + bx = c$ as well.

(e) L_e: a line passing through $(-1, 2)$ and $(-1, -3)$.
Here if we try to find m, we get division by zero. This is because the line vertical.
Its equation is $x = -1$, since the x coordinate of the two points is -1. This is already in the form $ax + by = c$ as well.



#2. Consider the line with equation $y = -3x - 1$.
(a) Find its intersections with the axes.
The intersection with the y axis is at $y = -1$.
For the intersection with the x axis, set $0 = -3x - 1$, so $x = -\frac{1}{3}$.
(b) Find a point which lies on this line and isn't on one of the axes.
We can just take any x value other than 0 or $-\frac{1}{3}$. For example, if $x = 1$, $y = -4$, so a point would be $(1, -4)$.
(c) Does the point $(-2, -7)$ lie on this line?
Substituting $x = -2$, we get $y = -3(-2) - 1 = 5$, which is not -7, so the point isn't on the line.

#3. Find the equations of the lines shown in the following diagram. Express each one in both forms, if possible.
l_1 is $y = 3$
l_2 is $y = \frac{1}{2}x + 1$, or $-x + 2y = 2$.
l_3 is $x = -2$
l_4 is $y = -3x - 3$, or $3x + y = -3$
l_5 is $y = 5x - 10$, or $5x - y = 10$