Practice for the Nov. 29th midterm.

Mr. Neeman. 10A, November 24, 2011

#1. Consider the function $f(x) = \frac{1}{2+x}$.
(a) Find its inverse function, f^{-1}.
We set $y = \frac{1}{2+x}$ and solve for x:
$(2 + x)y = 1$
$2 + x = \frac{1}{y}$
$x = \frac{1}{y} - 2$
Therefore, $f^{-1}(y) = \frac{1}{y} - 2$
(b) Find $f^{-1}(-1)$.
$f^{-1}(-1) = \frac{1}{-1} - 2 = -3$

#2. Consider the function $f(x) = \frac{1}{2}x^5$.
(a) Find its inverse function, f^{-1}.
We set $y = \frac{1}{2}x^5$ and solve for x:
$2y = x^5$
$\sqrt[5]{2y} = x$
Therefore, $f^{-1}(y) = \sqrt[5]{2y}$
(b) Find $f^{-1}(16)$.
$f^{-1}(16) = \sqrt[5]{2(16)} = \sqrt[5]{32} = 2$

#3. For each of the following pairs of functions, say whether or not they're inverses.
(a) $f(x) = 5x - 2$ and $g(x) = -5x + 2$
We can check whether $fg(x) = x$:
$fg(x) = f(g(x)) = f(-5x + 2) = 5(-5x + 2) - 2 = -25x + 10 - 2 = -25x + 8$
This isn't the same as x, so they're not inverses.
(b) $f(x) = x^3 + 1$ and $g(x) = \sqrt[3]{x - 1}$
We can check whether $fg(x) = x$:
$fg(x) = f(g(x)) = f(\sqrt[3]{x - 1}) = (\sqrt[3]{x - 1})^3 + 1 = x - 1 + 1 = x$
So they are inverses.
(c) $f(x) = \frac{2}{x} - 1$ and $g(x) = \frac{x}{2} + 1$
We can check whether $fg(x) = x$:
$$fg(x) = f(g(x)) = f(\tfrac{x}{2} + 1) = \frac{2}{\frac{x}{2} + 1} - 1 = \frac{2}{\frac{x+2}{2}} = \frac{4}{x + 2}$$
This isn't the same as x, so they're not inverses.

#4. Suppose L_1 has equation $y = \frac{1}{2}x - 1$
(a) Find the equation of the line, L_2, which is parallel to L_1 and passes through the point $(-1, -3)$.
The gradient of L_1 is $\frac{1}{2}$, so L_2 will also have gradient $\frac{1}{2}$. Substituting the point $(-1, -3)$, we get:
$-3 = \frac{1}{2}(-1) + b$
$-3 + \frac{1}{2} = b$
$-\frac{5}{2} = b$
So it's $y = \frac{1}{2}x - \frac{5}{2}$

(b) Find the equation of the line, L_3, which is perpendicular to L_1 and passes through the point $(5, -\frac{7}{2})$.

It will have gradient $-\dfrac{1}{\frac{1}{2}} = -2$. Substituting in $(5, -frac72)$, we get:

$-\frac{7}{2} = -2(5) + b$
$10 - \frac{7}{2} = b$
$\frac{13}{2} = b$

So it's $y = -2x + \frac{13}{2}$

(c) Find the point of intersection between L_1 and L_3.

The equations are $y = \frac{1}{2}x - 1$
and $y = -2x + \frac{13}{2}$

Substituting the first into the second, we get
$\frac{1}{2}x - 1 = -2x + \frac{13}{2}$
$\frac{5}{2}x = \frac{15}{2}$
$x = 3$

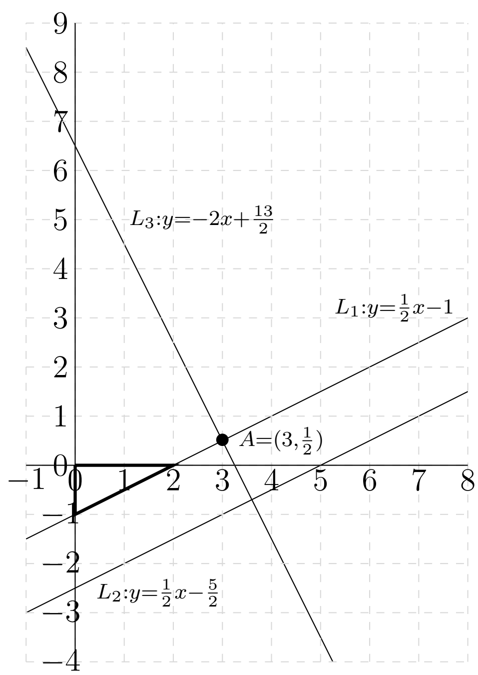
We then find y by substituting into one of the two, for example, the second:
$y = -2x + \frac{13}{2} = -2(3) + \frac{13}{2} = \frac{1}{2}$

So their point of intersection is $(3, \frac{1}{2}$.

(d) Find all the intersections of these lines with the axes.

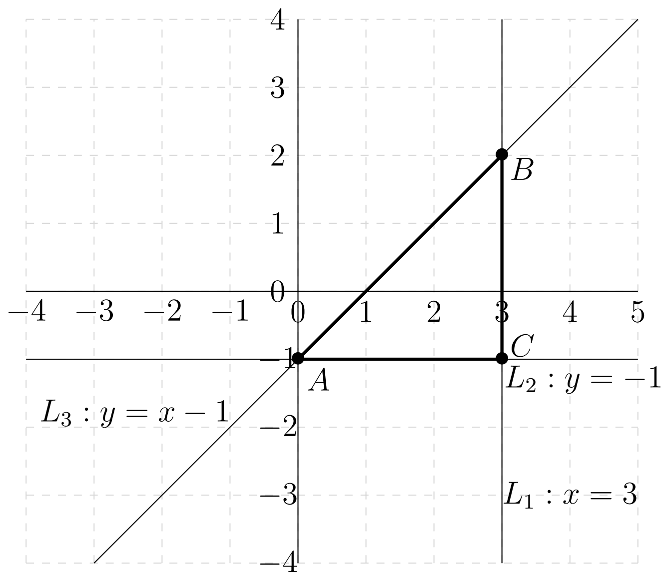
L_1 and the x axis: $y = 0$, which gives $x = 2$.
L_1 and the y axis: $x = 0$, which gives $y = -1$.
L_2 and the x axis: $y = 0$, which gives $x = 5$.
L_2 and the y axis: $x = 0$, which gives $y = -\frac{5}{2}$.
L_3 and the x axis: $y = 0$, which gives $x = \frac{13}{4}$.
L_3 and the y axis: $x = 0$, which gives $y = \frac{13}{2}$.

(e) Sketch a diagram representing these lines and all the intersections found in (c) and (d).



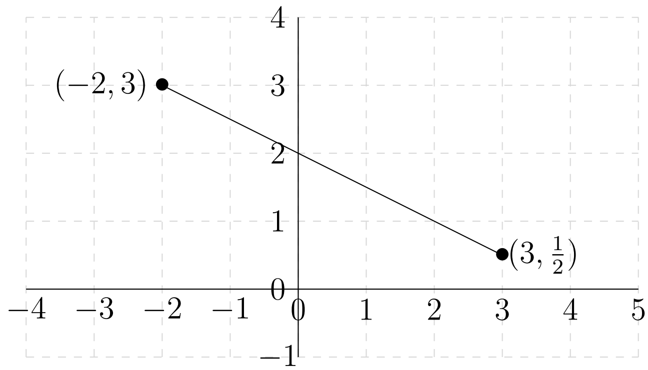
(f) Find the area of the triangle formed by the line L_1 and the axes.
The triangle is shown in bold in the diagram. We can see it's base is 2 and its height is 1. So the area is 1.

#5. (a) Find the equation of L_1, the vertical line which passes through the point $(3, 1)$.
$x = 3$
(b) Find the equation of a line, L_2, which is perpendicular to L_1 and which passes through the point $(-2, -1)$.
This line would be horizontal, and given the point it passes through the equation must be $y = -1$.
(c) Find the point of intersection of L_1 and L_2.
They intersect when $x = 3$ and $y = -1$, which is the point $(3, -1)$
(d) Sketch a diagram showing these lines and L_3, the line whose equation is $y = x - 1$



(e) Find the area of the triangle formed by the lines L_1, L_2, and L_3.
The triangle is shown in bold. We can see it's base is 3 and it's height is also 3, so the area is $\frac{9}{2}$.
(f) Find the midpoints of the triangle's three sides.
For convenience, we'll label the vertices as A, B, and C as in the diagram.
The midpoint of AB is $(\frac{3}{2}, \frac{1}{2})$.
The midpoint of BC is $(3, \frac{1}{2})$.
The midpoint of AC is $(\frac{3}{2}, -1)$.
(g) Find the triangle's perimeter.
We know the base and height are each of length 3. The length of the hypoteneuse is therefore $\sqrt{3^2 + 3^2} = 3\sqrt{2}$.
So the perimeter is $6 + 3\sqrt{2}$, which is approximately 10.2.

#6. Consider the function $f : [-2,3] \to \mathbb{R}$, with $f(x) = -\frac{1}{2}x + 2$.
(a) Find the function's range.
First, we find the images of the domain's endpoints:
$f(-2) = -\frac{1}{2}(-2) + 2 = 1 + 2 = 3$
$f(3) = -\frac{1}{2}(3) + 2 = \frac{1}{2}$
So the range is $[\frac{1}{2}, 3]$.
(b) Sketch the function's graph, labelling any intersections with the axes.
There is no intersection with the x axis, since the range is an interval which doesn't include zero (so $y = 0$ has no solutions, so 0 has no preimages).
The intersection with the y axis is when $y = f(0) = 2$.



(c) Is the function injective?
Yes.
(d) Is the function surjective?
No.
(e) Is the function bijective?
No. (f) What is the function's monotonicity?

#7. Consider the function $f : \{-1, 0, 1, 4\} \to \{-2, 0, 1, 2, 6, \}$, with $f(-1) = 0$, $f(0) = 2$, $f(1) = 6$, and $f(4) = -2$.
(a) Find the image of 0.
2.
(b) Find any preimages of 0.
-1.
(c) Find the function's range.
$\{-2, 0, 2, 6\}$.
(d) Is the function injective?
Yes.
(e) Is the function surjective?
No.
(f) Is the function bijective?
No.