Review: factorization (continued)

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Special product formulas

Remember, from last time:

$(a+b)^2 = a^2 + 2ab + b^2$
$(a-b)^2 = a^2 - 2ab + b^2$
$(a+b)(a-b) = a^2 - b^2$ (also known as the difference of squares product).

Note that there is no way to factorize $a^2 + b^2$ (there is no "sum of squares" formula).

Factorization: sum and difference of cubes

To these special product formulas, we can add a couple which are slightly more complicated:

$a^3 - b^3 = (a^2 + ab + b^2)(a - b)$ (difference of cubes)
$a^3 + b^3 = (a^2 - ab + b^2)(a + b)$ (sum of cubes)

We can verify these by multiplying out and cancelling terms. Just like with the special product formulas we had before, these can come in handy in factorization. You just need to be able to notice when you have a sum of cubes or a difference of cubes. Sometimes you would need to use the cubic root. For example:

$x^3 - 3 = (x^3 - (\sqrt[3]{3})^3) = (x^2 + \sqrt[3]{3}x + \sqrt[3]{3}^2)(x - \sqrt[3]{3})$

Note that if you substitute $-b$ for b in one of these formulas you will end up with the other (e.g. if you simplify $a^3 + (-b)^3 = (a^2 - a(-b) + (-b)^2)(a - b)$ you will get the difference of cubes formula). So you only need to remember one of them and you can get the other from it this way.

Completing the square

Completing the square is an odd-one-out in a way. It does involve factorization, but it doesn't right away give you a factorized expression which equals your original expression. We can start with an example:

Say you have the quadratic expression $x^2 + 4x - 5$. We could factorize this by inspection, noticing that 5 is only divisible by 1 and by itself, and say $x^2 + 4x - 5 = (x - 5)(x + 1)$. However, for other quadratic expressions this might not be possible, especially when fractions are involved. Completing the square is another possibility. What we do is we focus on the x^2 and x terms, and see what it would take to get them into a perfect square, let's say $(x+k)^2$. But $(x + k)^2 = x^2 + 2kx + k^2$. We then proceed to match up the terms, so we see that $2kx$ has to correspond to $4x$, and so k has to be 2. Therefore, what we're missing is $k^2 = 4$. So we add and subtract 4 to our expression:

$x^2 + 4x - 5 = x^2 + 4x + 4 - 4 - 5 = (x^2 + 4x + 4) - 4 - 5 = (x + 2)^2 - 9$

What we've done up to here is called "completing the square", because we added what was necessary to get a perfect square we can factorize using the special product formula. Now, from here we can factorize further, using the difference of squares formula:

$(x + 2)^2 - 9 = ((x + 2) + 3)((x + 2) - 3) = (x + 5)(x - 1)$.

So we seem to have gotten to the same answer in a roundabout way. The next example shows a case when this might be a good idea.

Let's complete the square for x^2+6x+2. In this case we can't simply factorize by inspection. So we proceed as before. This time, since the coefficient of x is 6, we will need to add and subtract 9 (since $9 = 3^2$ and 6 is twice 3).

$x^2 + 6x + 2 = x^2 + 6x + 9 - 9 + 2 = (x + 3)^2 - 7$

We now see why inspection wasn't possible: 7 is not a perfect square, so if we use the difference of squares formula our answer is going to involve square roots

$(x+3)^2 - 7 = (x+3)^2 - (\sqrt{7})^2 = ((x+3) + \sqrt{7})((x+3) - \sqrt{7}) = (x+3+\sqrt{7})(x+3-\sqrt{7})$.

One can always complete the square when presented with a quadratic expression. However, one sometimes can't then go on and factorize this way. For example:

$x^2 + 2x + 5 = x^2 + 2x + 1 - 1 + 5 = (x + 1)^2 + 4$

But this is a sum of squares, which we can't factorize. So filling in the square is also useful as a diagnostic tool. After you fill in a square, if the term which is left over is negative (as in the first two examples) you can use the difference of square formula. If it's positive, you can't factorize the expression. If the leftover term is 0, that means your expression was already a perfect square (e.g. $(x^2 + 6x + 9)$) so there wasn't any completing to be done.

So far, however, we only looked at cases when the coefficient of x^2 was 1. If, instead, it's a perfect square, it's not much harder. For example:

$9x^2 + 12x - 1 = 9x^2 + 12x + 4 - 4 - 1 = (3x + 2)^2 - 5$

We can then use the difference of squares formula to get $(3x + 2 + \sqrt{5})(3x + 2 - \sqrt{5})$.

If the coefficient of x^2 is not a perfect square things get messy. But they can be worked through slow but steady by factoring out the coefficient and then completing the square. For example:
$2x^2 + 3x - 8 = 2(x^2 + \frac{3}{2}x - 4) = 2(x^2 + \frac{3}{2}x + \frac{9}{16} - \frac{9}{16} - 4) = 2((x + \frac{3}{4})^2 - \frac{73}{16})$
$= 2(x + \frac{3}{4} + \frac{\sqrt{73}}{4})(x + \frac{3}{4} - \frac{\sqrt{73}}{4})$
$= 2(x + \frac{3+\sqrt{73}}{4})(x - \frac{3+\sqrt{73}}{4})$
$= \frac{1}{8}(4x + 3 + \sqrt{73})(4x - 3 - \sqrt{73})$
Note regarding this example: any of the three ways of writing it would be fine, I don't mean to imply the last one is better than the others, I just wanted to include the different ways.

If the coefficient of x^2 is negative, we can't get it to a perfect square since perfect squares are 0 or positive. So instead we get it to the negative of a perfect square. For example:

$-x^2 + 2x - 3 = -(x^2 - 2x + 3) = -(x^2 - 2x + 1 - 1 + 3) = -((x - 1)^2 + 2) = -(x - 1)^2 - 2$

We can't factorize this further, but we were able to complete the square.

Exercises

Factorize each of the following as far as you can. You may use any of the methods you're familiar with (including inspection).

#E1. $4x^2 - y^2z^2$
#E2. $2x^2 + 12x + 18$
#E3. $2x^3 + 54$
#E4. $2ax^2 + a^2x^2 + x^2$
#E5. $x^2 + 2x - 3$
#E6. $y^2 - 2a^2$

Complete the square for each of these expressions (even if you know how to factorize it another way, so you get practice in completing squares). Then factorize the resulting expression if possible.
#E7. $x^2 - 8x + 17$
#E8. $2x^2 + 5x - 3$
#E9. $-x^2 + 4x + 2$
#E10. $4x^2 + 10x - 5$

Solutions

#E1. $4x^2 - y^2z^2 = (2x + yz)(2x - yz)$
#E2. $2x^2 + 12x + 18 = 2(x^2 + 6x + 9) = 2(x + 3)^2$
#E3. $2x^3 + 54 = 2(x^3 + 27) = 2(x^2 + 3x + 9)(x - 3)$
#E4. $2ax^2 + a^2x^2 + x^2 = x^2(2a + a^2 + 1) = x^2(a + 1)^2$
#E5. $x^2+2x-3 = x^2+2x+1-1-3 = (x+1)^2-4 = (x+1+2)(x+1-2) = (x+3)(x-1)$
(#E5 can also be done by inspection)
#E6. $y^2 - 2a^2 = y^2 - (\sqrt{2}a)^2 = (y + \sqrt{2}a)(y - \sqrt{2}a)$

#E7. $x^2 - 8x + 17 = x^2 - 8x + 16 - 16 + 17 = (x - 4)^2 + 1$ (can't be factorized further)

#E8. $2x^2 + 5x - 3 = 2(x^2 + \frac{5}{2}x - \frac{3}{2}) = 2(x^2 + \frac{5}{2}x + \frac{25}{16} - \frac{25}{16} - \frac{3}{2})$
$= 2((x + \frac{5}{4})^2 - \frac{49}{16}) = 2(x + \frac{5}{4})^2 - \frac{49}{8}$ (can't be factorized further)

#E9. $-x^2 + 4x + 2 = -(x^2 - 4x - 2) = -(x^2 - 4x + 4 - 4 - 2)$
$= -((x - 2)^2 - 6) = 6 - (x - 2)^2 = (\sqrt{6} + 2 - x)(\sqrt{6} - 2 + x)$

#E10. $4x^2 + 10x - 5 = (2x^2) + 5(2x) - 5 = (2x^2) + 5(2x) + \frac{25}{4} - \frac{25}{4} - 5$
$= (2x + \frac{5}{4})^2 - \frac{45}{4}$
$= (2x + \frac{5}{4} + \sqrt{\frac{45}{4}})(2x + \frac{5}{4} - \sqrt{\frac{45}{4}})$
$= (2x + \frac{5+3\sqrt{5}}{2})(2x + \frac{5-3\sqrt{5}}{2})$