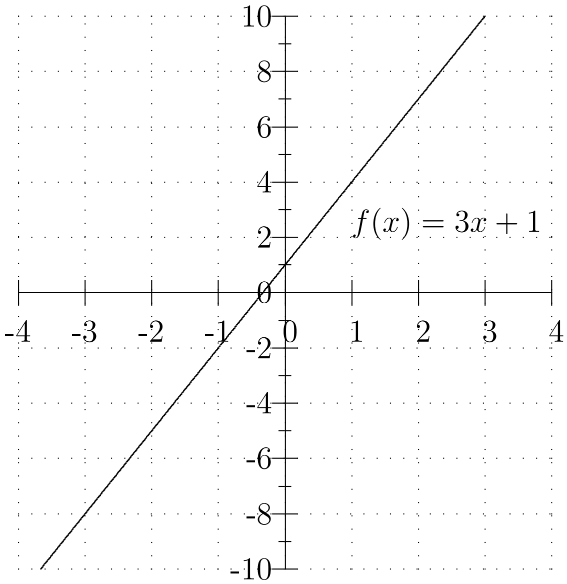
Inverse functions
Mr. Neeman, 10A. October 26, 2011

Recall that for a function to be **bijective** means that each element of the codomain has exactly one preimage. For example, $f(x) = 3x + 1$, whose graph is shown below, is bijective.



We've already seen how to find preimages: to find the preimage of a, we set $f(x)$ equal to a, and solve the resulting equation to find the value of x.

E.g. The preimage of 13 if $f(x) = 3x + 1$:
$3x + 1 = 13$
$3x = 12$
$x = 4$

This can be done for any value of a. **The inverse function of f, denoted f^{-1}, is the function which maps every number to its preimage under f.** To make things a little clearer, I will use y here, but remember that x and y are just dummy variables. This can be written:

> $f^{-1}(y) =$ the preimage of y under f
>
> Putting it differently: $f^{-1}(y) = x$ if and only if $y = f(x)$.

One way of thinking about it is that the inverse function is like finding the preimage, but doing it for any value rather than for a particular one. In fact, this is how we actually find the inverse function.

E.g. #1. Suppose $f(x) = 3x + 1$. Then to find f^{-1}, we set $f(x)$ equal to y and solve for x:
$y = 3x + 1$
$y - 1 = 3x$
$\frac{y-1}{3} = x$
Therefore, $f^{-1}(y) = \frac{y-1}{3}$
We can check this fits what we had before, since $f^{-1}(13) = \frac{13-1}{3} = 4$, and we had earlier found 4 was the preimage of 13.
This tells us that, whatever y is, the preimage of y under f will be $\frac{y-1}{3}$.

Note that for f to have an inverse it has to be bijective, because if some number had no preimage or more than one preimage, then f^{-1} wouldn't be a function (it would give zero or more than one output for some input).

E.g. #2. Find the inverse function of $f(x) = \frac{1}{2}x^3$.
As before, we go about as if finding a preimage, but using y instead of a particular value:
$y = \frac{1}{2}x^3$
$2y = x^3$
$\sqrt[3]{2y} = x$
Therefore, the preimage of y is $\sqrt[3]{2y}$, so that $f^{-1}(y) = \sqrt[3]{2y}$

Inverse functions and composition

Suppose f is a bijective function, and f^{-1} is its inverse function.
That tells us that:

$f^{-1}(y) = x$ if and only if $y = f(x)$
Suppose that we have some x and y such that $y = f(x)$, then we know $f^{-1}(y) = x$. We can then subsitute $f(x)$ instead of y, giving:
$f^{-1}(f(x)) = x$
Put differently, remembering the definition of composition:
$f^{-1}f(x) = x$.
Likewise, we can substitute $f^{-1}(y)$ for x on the right, to get:
$y = f(f^{-1}(y))$
This is the same as saying:
$ff^{-1}(y) = y$.

But now we can remember that x and y are just dummy variables, so it means $ff^{-1}(x) = x$ (not the same x of course, but any x in the range of f this time).

Therefore: $\boxed{\begin{array}{l} f^{-1}f(x) = x \\ ff^{-1}(x) = x \end{array}}$

Let's check that this works with our previous examples

E.g. #1. If $f(x) = 3x + 1$, then $f^{-1}(x) = \frac{x-1}{3}$.
So we have $f^{-1}f(x) = f^{-1}(f(x)) = f^{-1}(3x+1) = \frac{(3x+1)-1}{3} = \frac{3x}{3} = x$,
and $ff^{-1}(x) = f(f^{-1}(x) = f(\frac{x-1}{3}) = 3\frac{x-1}{3} + 1 = x - 1 + 1 = x$.
So we see these are satisfied.

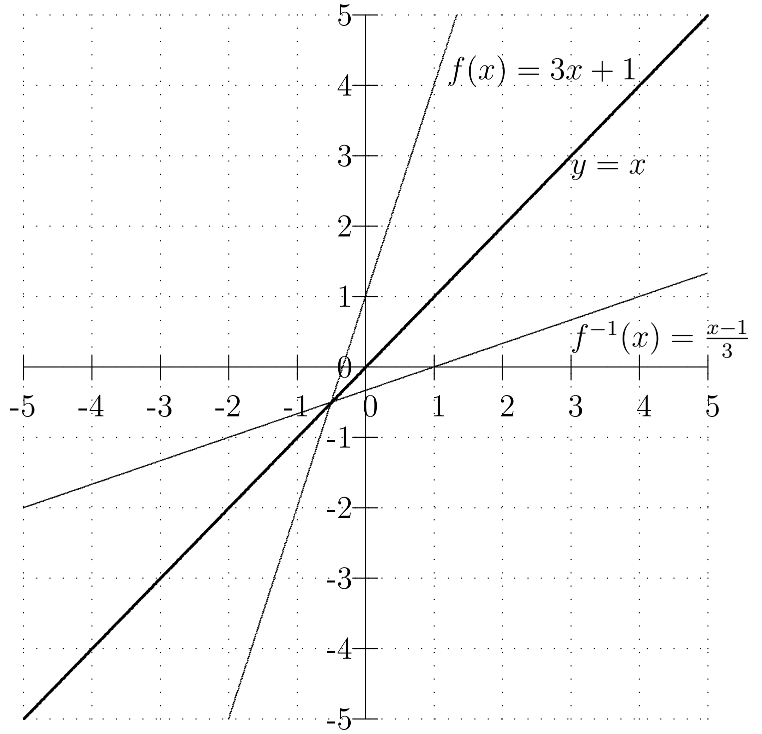
E.g. #2. If $f(x) = \frac{1}{2}x^3$, then $f^{-1}(x) = \sqrt[3]{2x}$.
So we have $f^{-1}f(x) = f^{-1}(f(x)) = f^{-1}(\frac{1}{2}x^3) = \sqrt[3]{2(\frac{1}{2}x^3)} = \sqrt[3]{x^3} = x$,
and $ff^{-1}(x) = f(f^{-1}(x) = f(\sqrt[3]{2x}) = \frac{1}{2}(\sqrt[3]{2x})^3 = \frac{1}{2}(2x) = x$.
So we see these are satisfied.

This is a good way to check whether two functions are inverses:
Suppose f and g are two bijective functions. Then they're inverses of each other if and only if $fg(x) = x$.

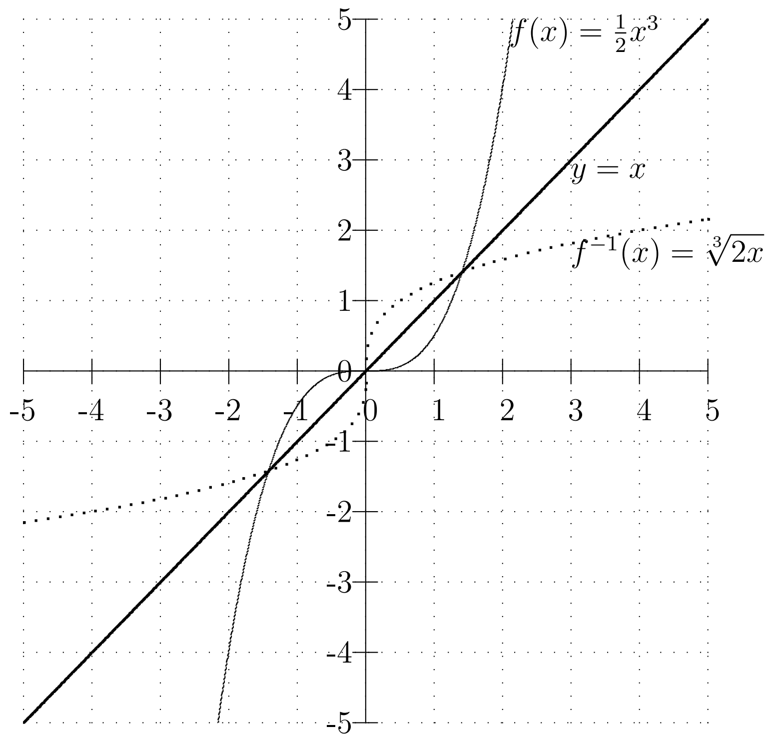
The other way to check is just to take f, find its inverse function, and see whether it's the same as g.

Graphs of inverse functions

If f is a bijective function, so that it has an inverse, f^{-1}, the graph of f^{-1} is the graph of f reflected around the line $y = x$. This is shown in the diagram below for our first example.



The diagram below shows this for our second example.



This makes sense because finding the inverse function is like switching x and $f(x)$, so it's like switching x with y.

Practice exercises

#P1. Find the inverse function of each of the following:
(a) $f(x) = \frac{x}{4} - 6$
(b) $f(x) = \frac{1}{x}$
(c) $f(x) = -x + 1$

#P2. For each of the following pairs of functions, check whether they are inverses of each other:
(a) $f(x) = 7x - 1$ and $g(x) = 7x + 1$
(b) $f(x) = x - 6$ and $g(x) = x + 6$
(c) $f(x) = x^3 - 3$ and $g(x) = \sqrt[3]{x+3}$

Homework for Monday, Oct. 31st

#1. Find the inverse function of each of the following:

(a) $f(x) = 10 - x$

(b) $f(x) = \frac{5x+12}{2}$

(c) $f(x) = \frac{1}{x-3}$

(d) $f(x) = \frac{1}{x^3}$.

#2. For each of the following pairs of functions, check whether they're inverses:

(a) $f(x) = \frac{1}{2x}$ and $f(x) = 2x$

(b) $f(x) = 2x^5$ and $g(x) = \frac{1}{2}\sqrt[5]{x}$

(c) $f(x) = 6 - 2x$ and $g(x) = -\frac{1}{2}y + 3$

Solutions for practice exercises

#P1. Find the inverse function of each of the following:

(a) $f(x) = \frac{x}{4} - 6$
$y = \frac{x}{4} - 6$
$y + 6 = \frac{x}{4}$
$4y + 24 = x$
Therefore, $f^{-1}(y) = 4y + 24$

(b) $f(x) = \frac{1}{x}$
$y = \frac{1}{x}$
$xy = 1$
$x = \frac{1}{y}$
Therefore, $f^{-1}(y) = \frac{1}{y}$. Notice that this is the same function as f. So f is actually its own inverse function. This is similar to how 1 is its own reciprocal ($\frac{1}{1} = 1$), or how -1 is its own reciprocal ($\frac{-1}{1} = -1$).

(c) $f(x) = -x + 1$
$y = -x + 1$
$x + y = 1$
$x = 1 - y$
Therefore, $f^{-1}(y) = 1 - y$.
Notice that this function is also its own inverse: $f^{-1}(y) = -y + 1$, so that $f^{-1}(x) = -x + 1$.

#P2. For each of the following pairs of functions, check whether they are inverses of each other:

(a) $f(x) = 7x - 1$ and $g(x) = 7x + 1$
$fg(x) = f(7x + 1) = 7(7x + 1) - 1 = 49x + 7 - 1 = 49x + 6$, which isn't x. So they're not inverses

(b) $f(x) = x - 6$ and $g(x) = x + 6$
$fg(x) = f(x + 6) = (x + 6) - 6 = x$, so they are inverses.

(c) $f(x) = x^3 - 3$ and $g(x) = \sqrt[3]{x + 3}$
$fg(x) = f(\sqrt[3]{x + 3}) = (\sqrt[3]{x + 3})^3 - 3 = x + 3 - 3 = x$, so they are inverses.