Quadratic inequalities

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Quadratic inequalities are like quadratic equations but with $\leq, <, \geq$, or $>$ instead of an equals sign. To solve quadratic inequalities, we proceed similarly to how we would in solving a quaratic equation. However, we must remember that if we divide or multiply by a negative number then the inequality sign has to be flipped. As usual, it's safe to just avoid multiplying or dividing by negative numbers.

When solving quadratic equations, we are either able to factorize the quadratic expression, or we have a quadratic expression which can't be factorized.

Case 1: a quadratic which can't be factorized

E.g. Solve the inequality $x^2 + 2x + 4 < 0$. In other words, we have to find the values of x which make $x^2 + 2x + 4$ negative. As if we were solving the corresponding equation ($x^2 + 2x + 4 = 0$) we can fill in the square:

$(x + 1)^2 + 3 < 0$

Now, intuitively, we can see that the first term can never be negative because it's a square, so there's no way that adding 3 to it will give us a negative number. Therefore, the inequality has no solutions.

Another way to think about it is that the corresponding equation has no solutions (i.e. there are no values of x for which $(x + 1)^2 + 3 = 0$. So the expression is either always positive or always negative. We can try out a value, say when $x = 0$, and we find that it's positive. So that means it's always positive, and never negative. So there are no values of x for which $x^2 + 2x + 4 < 0$.

E.g. Solve the inequality $x^2 + 2x + 4 \geq 0$.

Given the analysis above, we know that $x^2 + 2x + 4$ is always positive, whatever x is. So the inequality is satisfied for all x. Therefore the solution is all real numbers.

E.g. Solve the inequality $-x^2 + 4x - 20 < 0$.

We pass everything to the other side: $0 < x^2 - 4x + 20$ and complete the square:

$0 < (x - 2)^2 + 16$

The right hand side is a sum of squares, which can't be factorized. Now, the right hand side is always positive, since 16 is positive and $(x - 2)^2$ can't be negative. So the inequality is satisfied for any value of x, so that all real numbers are solutions.

Case 2: quadratics which can be factorized

Subcase A: perfect squares

E.g. $x^2 + 6x + 9 > 0$.

We factorize the left hand side: $(x + 3)^2 > 0$

Now, $(x + 3)^2$ is zero when $x = -3$, and is positive otherwise. Therefore the inequality is satisfied for all real numbers except $x = 3$.

E.g. $x^2 + 6x + 9 \leq 0$.

We factorize the same way. As before, $(x + 3)^2$ is zero when $x = -3$ and positive otherwise. So it's less than or equal to 0 just when $x = -3$. So our answer is $x = -3$.

E.g. $x^2 + 6x + 9 < 0$.

We factorize the same way. As before, $(x + 3)^2$ is zero when $x = -3$ and positive otherwise. So it can never be negative, so that the inequality has no solutions.

Subcase B: quadratics which factorize into a product of two different terms

E.g. $x^2 + 2x - 12 \geq 0$
First, we factorize the left hand side"
$(x + 4)(x - 2) \geq 0$.
Now, we know that $(x + 4)(x - 2)$ is zero at $x = -4$ and at $x = 2$. What we'll do is carry out an analysis of its signs for other values of x. We have to remember that multiplying two positive numbers gives a positive one, multiplying two positive numbers also gives a positive one, and multiplying a positive by a negative gives a negative number.

	$x < -4$	$x = -4$	$-4 < x < 2$	$x = 2$	$x > 2$
$(x + 4)$	negative	0	postive	postive	positive
$(x - 2)$	negative	negative	negative	0	positive
$(x + 4)(x - 2)$	positive	0	negative	0	positive

Now, we're looking for where $(x + 4)(x - 2) \geq 0$. This is in all the columns except the one for $-4 < x < 2$, because there it's negative. Therefore, our answer is $x \leq -4$ and $x \geq 2$.

E.g. $2x^2 + 2x - 4 < 0$.
Again, we factorize:
$2(x - 1)(x + 2) < 0$
And we draw a table again. The key values are $x = 1$ and $x = -2$, since that's where our quadratic becomes zero.

	$x < -2$	$x = -2$	$-2 < x < 1$	$x = 1$	$x > 1$
$(x - 1)$	negative	negative	negative	0	positive
$(x + 2)$	negative	0	positive	positive	positive
$2(x - 4)(x + 4)$	positive	0	negative	0	positive

Now, we're looking for where it's negative, which is only for $-2 < x < 1$.
E.g. $-x^2 - 4x - 2 \geq 0$
To make things simpler, we move everything to the right hand side, so that x^2 has a positive coefficient:
$0 \geq x^2 + 4x + 2$
To factorize, we can first complete the square:
$0 \geq (x + 2)^2 - 2$
$0 \geq (x + 2 + \sqrt{2})(x + 2 - \sqrt{2})$
From here, we proceed as before with a table:

	$x < -2 - \sqrt{2}$	$x = -2 - \sqrt{2}$	$-2 - \sqrt{2} < x < -2 + \sqrt{2}$	$x = -2 + \sqrt{2}$	$x > -2 + \sqrt{2}$
$(x + 2 + \sqrt{2})$	negative	0	positive	positive	positive
$(x + 2 - \sqrt{2})$	negative	negative	negative	0	positive
$(x + 2 + \sqrt{2})(x + 2 - \sqrt{2})$	positive	0	negative	0	positive

We're looking for $0 \geq (x + 2 + \sqrt{2})(x + 2 - \sqrt{2})$, which means that
$-2 - \sqrt{2} \leq x \leq -2 + \sqrt{2}$

Practice exercises

Solve the following inequalities:

#P1. $2x^2 + 3x + 1 \leq 0$
#P2. $x^2 - 8x + 16 \leq 0$
#P3. $x^2 + 4x + 1 > 0$
#P4. $x^2 + 4x + 4 < 0$
#P5. $-2x^2 + 12x + 3 \geq 0$

Homework

Solve the following inequalities:

#H1. $x^2 - 8x + 16 \geq 0$
#H2. $x^2 + 2x - 15 < 0$
#H3. $2x^2 + 5x - 3 \leq 0$
#H4. $-x^2 + 2x - 1 > 0$
#H5. $x^2 + 4x - 2 \geq 0$
#H6. $2x^2 + 8x + 12 \leq 0$

Solutions for practice exercises

#P1. $2x^2 + 3x + 1 \leq 0$
$(2x+1)(x+1) \leq 0$
The key values are $x = -\frac{1}{2}$ and $x = -1$

	$x < -1$	$x = -1$	$-1 < x < -\frac{1}{2}$	$x = -\frac{1}{2}$	$x > -\frac{1}{2}$
$(x+1)$	negative	0	postive	postive	positive
$(2x+1)$	negative	negative	negative	0	positive
$(x+1)(2x+1)$	positive	0	negative	0	positive

So our answer is $-1 \leq x \leq -\frac{1}{2}$.

#P2. $x^2 - 8x + 16 \leq 0$
$(x-4)^2 \leq 0$
This is satisfied for $x = 4$ and no other values.

#P3. $x^2 + 4x + 1 > 0$
$(x+2)^2 - 3 > 0$
$(x + 2 + \sqrt{3})(x + 2 - \sqrt{3}) > 0$
The key values are $x = -2 - \sqrt{3}$ and $x = -2 + \sqrt{3}$. Doing the table, we find the inequality is satified when $x < -2 - \sqrt{3}$ and when $x > -2 + \sqrt{3}$.

#P4. $x^2 + 4x + 4 < 0$
$(x+2)^2 < 0$
This has no solutions.

#P5. $-2x^2 + 6x + 3 \geq 0$
$0 \geq 2x^2 - 6x - 3$
$0 \geq 2(x^2 - 6x - 3)$
$0 \geq 2((x-3)^2 - 12)$
$0 \geq 2(x - 3 - \sqrt{12})(x - 3 + \sqrt{12})$
So our key values are $x = 3 + \sqrt{12}$ and $x = 3 - \sqrt{12}$.
Doing the table, we find that the inequality is satisfied when $3 - \sqrt{12} \leq x \leq 3 + \sqrt{13}$.