Application of quadratic equations: free-fall

Mr. Neeman. 10A, August 5, 2011

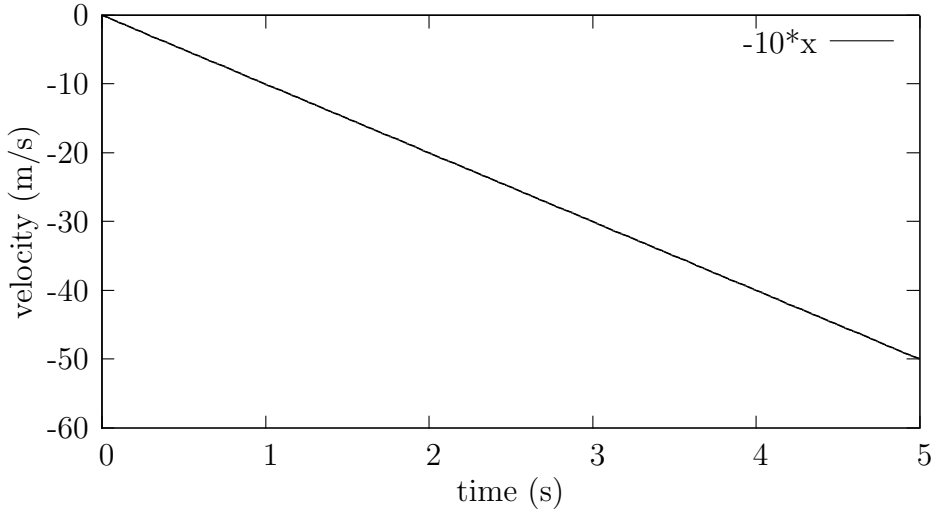
When we say displacement, we mean distance which has a direction. We will be using y to represent the height above ground-level, in meters. So, for example:
$y = 0$ means ground-level.
$y = 5$ means 5 meters above ground-level.
$y = -3$ means 3 meters below ground-level.

We will be considering the case of an object being dropped from rest at a given height. For example, say we hold a ball 10 meters above the ground and then simply drop it. It will accelerate due to gravity. However, gravity is *downwards*, and we took our y to be distance *above* ground-level. So the acceleration due to gravity will be negative. We will take an approximate value: -10 m/s^2.

What this means is that every second, the object's velocity will be 10 m/s less than before. Let's say that the time at which we drop is $t = 0$. Then we know that after 1 second, at $t = 1$, its velocity would be -10 m/s. After 4 seconds, at $t = 4$, its velocity would be -40 m/s, and so on. This is, of course, only valid so long as the object is in free-fall (i.e. until it hits an object such as the ground). We can express this using the following formula:

$v = -10t$, where v is the object's velocity, and t is the time since it was dropped.

Figure 1: Velocity in free fall



An example is illustrated in figure 1: an object is dropped from rest and is in free fall for 5 seconds. After those 5 seconds, its velocity is -50 m/s. Now, the distance traveled in those 5 seconds is given by the area between the graph and the t axis. This is a triangle, of base 5 seconds and height 50 m/s. So the area is 125 meters. This means that an object dropped from rest falls 125 meters in 5 seconds. More generally, at time t the base would be t seconds, and the height would be $10t$ m/s. So the area is $5t^2$. However, we should note that the objects are falling *down*, so that the displacement is $-5t^2$ from where we started. Say we started at height y_0, then we end up at $y = y_0 - 5t^2$. This is our general formula:

For an object dropped from rest: $y = y_0 - 5t^2$, where y_0 is the initial height, and t is the time elapsed since it was dropped.

We can then use this to work examples.

Example exercises

#1. A ball is dropped from a height of 30 meters. Find: (a) how high it will be after 1 second, (b) how long it will take to hit the ground, (c) when it would be at a height of 5 meters.
Solution: First of all, for all parts of the question, we're given the initial height is 30 meters. So we have $y_0 = 30$, and our formula becomes $y = 30 - 5t^2$.
(a) We want to find the height 1 second after it's dropped. So we substitute $t = 1$ into the equation and find y:
$y = 30 - 5(1)^2 = 25$. So it would be 25 meters above the ground.
(b) The ground-level is $y = 0$. So we want to figure out when y will be zero. We do this by substituting 0 for y and solving the equation:
$0 = 30 - 5t^2$
$5t^2 = 30$
$t^2 = 6$
$t = \sqrt{6}$
Now, normally, we would have two solutions: $\sqrt{6}$ and $-\sqrt{6}$. However, our formula is only valid for the time during which the ball is in free fall, which is starting at $t = 0$. So anything with t negative isn't meaningful in this context (before the ball is dropped, it wasn't in free-fall).
(c) We're looking for when the height is 5 meters. So we set $y = 5$ and solve for t:
$5 = 30 - 5t^2$
$5t^2 = 25$
$t^2 = 5$
$t = \sqrt{5}$.
So the ball will reach a height of 5 meters above ground-level $\sqrt{5}$ seconds after being dropped.

#2. A rock is dropped from a height of 64 meters. Find: (a) how high it will be after 2 seconds, (b) how long it will take to hit the ground, (c) when it would be at a height of 5 meters, (d) how high it will be after $\sqrt{10}$ seconds, (e–optional advanced) how long it would take to fall the last 5 meters before hitting the ground.
Solution: First of all, for all parts of the question, we're given the initial height is 64 meters. So we have $y_0 = 64$, and our formula becomes $y = 64 - 5t^2$.
(a) We want to find the height 2 seconds after it's dropped. So we substitute $t = 2$ into the equation and find y:
$y = 64 - 5(2)^2 = 44$. So it would be 44 meters above the ground.
(b) The ground-level is $y = 0$. So we want to figure out when y will be zero. We do this by substituting 0 for y and solving the equation:
$0 = 64 - 5t^2$
$5t^2 = 64$
$t^2 = \frac{64}{5}$

$t = \frac{8}{\sqrt{5}}$
(c) We're looking for when the height is 5 meters. So we set $y = 5$ and solve for t:
$5 = 64 - 5t^2$
$5t^2 = 59$
$t^2 = \frac{59}{5}$
$t = \sqrt{\frac{59}{5}}$ (So it reaches that height after $\sqrt{\frac{59}{5}}$ seconds).
(d) We want to find the height $\sqrt{10}$ seconds after it's dropped. So we substitute $t = \sqrt{10}$ into the equation and find y:
$y = 64 - 5(\sqrt{10})^2 = 14$. So it would be 14 meters above the ground.
(e) We already calculated above that it will be 5 meters high at time $t = \frac{59}{5}$ and will hit the ground at $t = \frac{8}{\sqrt{5}}$. The time it will take to travel the last 5 meters is the difference between these: $\frac{8}{\sqrt{5}} - \frac{59}{5}$.

Practice exercises (solutions on next page)

#P1. Suppose a ball is dropped from a height of 120 meters. Find (a) how long it will take to hit the ground, (b) how long it will take to get halfway to the ground, (c) how high above the ground it would be after 3 seconds, (d) when it would be at a height of 80 meters.
#P2. Suppose a ball is dropped from a height of 1 kilometer. Find (a) how long it will take to hit the ground, (b) how high it would be after 5 seconds, (c) when it would be halfway to hitting the ground.
#P3. Suppose a ball is dropped from a height of 1 meter. Find (a) how long it will take to hit the ground, (b) how high it would be after a quarter of a second, (c) when it would be two-thirds of the way to hitting the ground.

Homework for Monday

#H1. Suppose a ball is dropped from a height of 150 meters. Find (a) how long it will take to hit the ground, (b) how long it will take to get halfway to the ground, (c) how high above the ground it would be after 5 seconds, (d) when it would be at a height of 60 meters.
#H2. Suppose a ball is dropped from a height of 2 kilometer. Find (a) how long it will take to hit the ground, (b) how high it would be after 13 seconds, (c) when it would be at a height of 500 meters.
#H3. Suppose a ball is dropped from a height of 10 centimeters. Find (a) how long it will take to hit the ground, (b) how high it would be after a tenth of a second, (c) when it would be 8 centimeters above the ground.

Solutions for practice problems

#P1. Suppose a ball is dropped from a height of 120 meters. Find (a) how long it will take to hit the ground, (b) how long it will take to get halfway to the ground, (c) how high above the ground it would be after 3 seconds, (d) when it would be at a height of 80 meters.

The equation is $y = 120 - 5t^2$
(a) We need to solve $y = 0$, so $5t^2 = 120$, and $t^2 = 24$. So $t = \sqrt{24} = 2\sqrt{6}$ (seconds).
(b) Halfway to the ground means at a height of 60 meters in this case. So we need to solve $y = 60$: $60 = 120 - 5t^2$, so that $5t^2 = 60$ and $t^2 = 12$, and $t = \sqrt{12}$ (seconds).
(c) Here we have $t = 3$, so we substitute this into our equation: $y = 120 - 5(3)^2 = 120 - 45 = 75$ (meters).
(d) We need to solve $y = 80$, so $80 = 120 - 5t^2$ and $5t^2 = 40$. This means $t^2 = 8$, so $t = 2\sqrt{2}$ (seconds).

#P2. Suppose a ball is dropped from a height of 1 kilometer. Find (a) how long it will take to hit the ground, (b) how high it would be after 5 seconds, (c) when it would be halfway to hitting the ground.

The equation is $y = 1000 - 5t^2$, since a kilometer is 1000 meters.
(a) We need to solve $y = 0$, so $5t^2 = 1000$, and $t^2 = 200$. So $t = 10\sqrt{2}$ (seconds).
(b) Here we have $t = 5$, so we substitute this into our equation: $y = 1000 - 5(5)^2 = 1000 - 125 = 875$ (meters).
(c) Halfway to the ground means at a height of 500 meters in this case. So we need to solve $y = 500$: $500 = 1000 - 5t^2$, so that $5t^2 = 500$ and $t^2 = 100$, and $t = 10$ (seconds).

#P3. Suppose a ball is dropped from a height of 1 meter. Find (a) how long it will take to hit the ground, (b) how high it would be after a quarter of a second, (c) when it would be two-thirds of the way to hitting the ground.
The equation is $y = 1 - 5t^2$.
(a) We need to solve $y = 0$, so $5t^2 = 1$, and $t^2 = \frac{1}{5}$. So $t = \frac{1}{\sqrt{5}}$ (seconds).
(b) Here we have $t = \frac{1}{4}$, so we substitute this into our equation: $y = 1 - 5(\frac{1}{4})^2 = 1 - \frac{1}{16} = \frac{15}{16}$ (meters).
(c) Two-thirds of the way to the ground means at a height of $\frac{1}{3}$ meters in this case. So we need to solve $y = \frac{1}{3}$: $\frac{1}{3} = 1 - 5t^2$, so that $5t^2 = \frac{2}{3}$ and $t^2 = \frac{2}{15}$, and $t = \sqrt{\frac{2}{15}}$ (seconds).