Functions: substitution and composition
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Substitution

Recall that a function maps inputs to outputs. We usually represent the input as x. However, the input can be any number or algebraic expression (of course, if it's not in the domain the output won't be defined). We get the output by substituting the input into the function.

E.g. Suppose $f(x) = 3x^2$. We can substitute anything for x (what we substitute for x will depend on the reason why we're doing it). For example:
$f(y) = 3y^2$ (illustrating that x and y are just placeholders)
$f(z+1) = 3(z+1)^2$
$f\left(\frac{1}{x}\right) = 3\left(\frac{1}{x}\right)^2 = \frac{3}{x^2}$
And so on.

E.g. Suppose $f(x) = x + \frac{1}{x+1}$. Then
$f(z^2) = z^2 + \frac{1}{z^2+1}$
$f(y-1) = y - 1 + \frac{1}{(y-1)+1} = y - 1 + \frac{1}{y}$
$f(2x) = 2x + \frac{1}{2x+1}$
And so on.

Function composition

Notice that when we substitute something instead of x, we get a new expression. If what we substitute for x is itself in terms of x, then we get a new expression which is in terms of x.

This allows us to combine functions, in a way called **function composition**.

Definition: Suppose $f : A \to B$ and $g : B \to C$ are two functions. Then their composition, denoted as fg or also as $f \circ g$ is the function defined as follows:
$fg : A \to C$
$fg(x) = f(g(x))$

Often, A, B, and C will all be $\mathbb{R}$. In such cases, one can calculate both fg and gf, which usually won't be the same.

E.g. Suppose $f(x) = 3x^2$ and $g(x) = x + 4$. Find the functions fg and gf.
To find fg, we use the definition: $fg(x) = f(g(x))$
This means that we take $g(x)$, which is $x + 4$, and substitute it into f:
$fg(x) = f(x + 4)$
We then use the fact that $f(x) = 3x^2$:
$fg(x) = 3(x + 4)^2$

Similarly, to find gf, we use the definition: $gf(x) = g(f(x))$
This means that we take $f(x)$, which is $3x^2$, and substitute it into g:
$gf(x) = g(3x^2)$
We then use the fact that $g(x) = x + 4$:
$gf(x) = 3x^2 + 4$
Notice that fg and gf aren't the same function.

E.g. Find the mappings of the functions fg and gf if $f(x) = 3x - 1$ and $g(x) = \dfrac{1}{x+1}$

$fg(x) = f(g(x)) = f\left(\dfrac{1}{x+1}\right) = 3\dfrac{1}{x+1} - 1 = \dfrac{3}{x+1} - \dfrac{x+1}{x+1} = \dfrac{2-x}{x+2}$

$gf(x) = g(f(x)) = g(3x-1) = \dfrac{1}{(3x-1)+1} = \dfrac{1}{3x}$

A good way to remember which functions goes where is the following:
In $fg(x)$, the function closest to the x is the one that gets applied first, so first we apply g to x and then we apply f to the result $(g(x))$.

Practice exercises

#P1. Suppose $f(x) = \dfrac{1}{x} + 2x$ and $g(x) = 3x - 1$. Find:
(a) $f(x+2)$
(b) $fg(x)$
(c) $gf(x)$

#P2. Suppose $f(x) = x^2 - 1$ and $g(x) = \dfrac{1}{x-1}$. Find:
(a) $g(4z)$
(b) $f(2x+1)$
(c) $fg(x)$
(d) $gf(y)$
(e) $gf(3)$

Homework

#H1. Suppose $f(x) = 4x + 3$ and $g(x) = x^2 + 2x$. Find:
(a) $f(z^3)$
(b) $g(2y)$
(c) $fg(x)$
(d) $gf(x)$
(e) $gf(-1)$

#H2. Suppose $f(x) = x^2 - \dfrac{1}{x}$ and $g(x) = x - 2$. Find:
(a) $f(2)$
(b) $f(x+2x^2)$
(c) $fg(y)$
(d) $gf(x^2)$

Solutions for practice exercises

#P1. Suppose $f(x) = \frac{1}{x} + 2x$ and $g(x) = 3x - 1$. Find:

(a) $f(x+2) = \frac{1}{x+2} + 2(x+2) = \frac{1}{x+2} + 2x + 4$

(b) $fg(x) = f(g(x)) = f(3x-1) = \frac{1}{3x-1} + 2(3x-1) = \frac{1}{3x-1} + 6x - 2$

(c) $gf(x) = g(f(x)) = g\left(\frac{1}{x} + 2x\right) = 3\left(\frac{1}{x} + 2x\right) - 1 = \frac{3}{x} + 6x - 1$

#P2. Suppose $f(x) = x^2 - 1$ and $g(x) = \frac{1}{x-1}$. Find:

(a) $g(4z) = \frac{1}{4z-1}$

(b) $f(2x+1) = (2x+1)^2 - 1 = 4x^2 + 4x$

(c) $fg(x) = f(g(x)) = f\left(\frac{1}{x-1}\right) = \left(\frac{1}{x-1}\right)^2 - 1 = \frac{1}{(x-1)^2} - 1$

(d) $gf(y) = g(f(y)) = g(y^2 - 1) = \frac{1}{(y^2-1)-1} = \frac{1}{y^2-2}$

(e) $gf(3) = \frac{1}{3^2-2} = \frac{1}{7}$