Logic (part 2)

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Truth tables

Truth tables are a simple way to study the logical properties of propositions and the logical relations between several propositions. Given any proposition, one can construct a truth table for it. In order to do this, though, the proposition must first be written in logical notation.

Truth tables are useful for dealing with compound propositions. For example, if you wanted to know whether the sentence $\neg p \Rightarrow p$ is a contradiction, you could do this by means of a truth table. However, we first need to have some basic truth tables. The following embody the meaning of the operators.

Truth table for negation.

p	$\neg p$
T	F
F	T

What this table is saying is that there are two cases: p can be true or it can be false. If p is true, then $\neg p$ is false; and if p is false, then $\neg p$ is true. "True" and "false" are called **truth values**.

Truth table for the other operators. We could draw a truth table for each one. However, to save space, I will just give one table for all the other operators.

p	q	$p \wedge q$	$p \vee q$	$p \veebar q$	$p \Rightarrow q$	$p \Leftrightarrow q$
T	T	T	T	F	T	T
T	F	F	T	T	F	F
F	T	F	T	T	T	F
F	F	F	F	F	T	T

Let's examine this table:
1. There are four rows. This will always be the case when we have a compound proposition involving two simple propositions (p and q in this case). This is because there are 4 different possible combinations of true and false for p and q. To avoid messing up the 4 rows, the best way to start the table is to go "true, false, true, false" for the simple sentence on the right (q here), and then "true, true, false, false" for the simple sentence on the left (p here).
2. The third column, for $p \wedge q$, tells us that $p \wedge q$ is true if both p and q are true, and false otherwise. This is just embodying the ordinary meaning of "and".
3. The fourth column, for $p \vee q$, tells us that $p \vee q$ is true if either p or q is true (including if both are true, as this is an inclusive or). This is just embodying the ordinary inclusive sense of "or".
4. The fifth column, for $p \veebar q$, tells us that $p \veebar q$ is true if one of them is true and the other false. In other words, it's true if one of them is true but not both of them. This is the exclusive sense of "or".
5. The sixth column, for $p \Rightarrow q$, tells us that $p \Rightarrow q$ is true except if p is true and q is false. This is the one that gives people the most trouble, and, in fact, there are some interesting philosophical questions regarding its interpretation. One way to think about it is the following: what would it take to convince you that $p \Rightarrow q$ is false. For example, what would it take to convince you that "If it's raining, then there are clouds in the sky" is false? Well, you would have to witness it raining without there being clouds in the sky. In other words, to show that $p \Rightarrow q$ is false, you would need p to be true and q false.
6. The seventh column, for $p \Leftrightarrow q$ tells us that $p \Leftrightarrow q$ is true when p and q are both false and when they're both true; but is false when one of is true and the other false. This embodies the ordinary sense of "if and only if": it's true if they're either both true or both false.

Constructing truth tables

Suppose we want to do a truth table for a sentence, for example, $p \Rightarrow \neg q$. We follow the following steps:

1. Identify the simple propositions which appear in our compound proposition. (In our case, they're p and q.)
2. Construct the left side of the truth table to show all the possible combinations of true and false for the simple propositions. (In our case, this means four rows, as in the table on the previous page).

p	q
T	T
T	F
F	T
F	F

3. Build up the compound sentence step by step, representing each step as a column. You do not need to repeat the columns for p and q which are already on the left.

p	q	$\neg q$	$p \Rightarrow \neg q$
T	T		
T	F		
F	T		
F	F		

4. Fill in the table column by column, from left to right. For this you will have to be very familiar with the truth tables for the operators (what is on the previous page).

First:

p	q	$\neg q$	$p \Rightarrow \neg q$
T	T	F	
T	F	T	
F	T	F	
F	F	T	

Then:

p	q	$\neg q$	$p \Rightarrow \neg q$
T	T	F	F
T	F	T	T
F	T	F	T
F	F	T	T

This tells us that our propositions is true when p and q are both true (the first row) and false otherwise (the other three rows).

With more complicated propositions, the procedure is the same, but the number of columns can be larger. If we had a proposition with 3 simple propositions (e.g. p, q, and r) we would need to do 8 rows, but we won't be doing tables that big.

E.g. Construct a truth table for $\neg p \wedge \neg q$.

p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

E.g. Construct a truth table for $\neg(p \vee \neg q)$

p	q	$\neg q$	$p \vee \neg q$	$\neg(p \vee \neg q)$
T	T	F	T	F
T	F	T	T	F
F	T	F	F	T
F	F	T	T	F

Practice exercises

#P1. Draw truth tables for each of the following propositions:

(a) $\neg p \vee q$
(b) $\neg(p \Rightarrow q)$
(c) $p \Rightarrow (\neg p \vee q)$
(d) $p \wedge (q \Leftrightarrow \neg p)$
(e) $p \Rightarrow \neg p$
(f) $p \veebar (\neg q \vee p)$

Homework

#H1. Draw a truth table for each of the following sentences:

(a) $\neg p \wedge \neg q$
(b) $\neg q \Leftrightarrow (p \vee q)$
(c) $p \Rightarrow (\neg q \Rightarrow p)$
(d) $p \veebar (\neg p \Leftrightarrow q)$
(e) $\neg q \vee (q \Rightarrow \neg p)$

Solutions for practice exercises

#P1. Draw truth tables for each of the following propositions:

(a) $\neg p \vee q$

p	q	$\neg p$	$\neg p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

(b) $\neg(p \Rightarrow q)$

p	q	$p \Rightarrow q$	$\neg(p \Rightarrow q)$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

(c) $p \Rightarrow (\neg p \vee q)$

p	q	$\neg p$	$\neg p \vee q$	$p \Rightarrow (\neg p \vee q)$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

(d) $p \wedge (q \Leftrightarrow \neg p)$

p	q	$\neg p$	$q \Leftrightarrow \neg p$	$p \wedge (q \Leftrightarrow \neg p)$
T	T	F	F	F
T	F	F	T	T
F	T	T	T	F
F	F	T	F	F

(e) $p \Rightarrow \neg p$

p	$\neg p$	$p \Rightarrow \neg p$
T	F	F
F	T	T

(f) $p \veebar (\neg q \vee p)$

p	q	$\neg q$	$\neg q \vee p$	$p \veebar (\neg q \vee p)$
T	T	F	T	F
T	F	T	T	F
F	T	F	F	F
F	F	T	T	T