Quadratic equations

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<u>Solving quadratic equations</u>

We first consider three simple quadratic equations:
(1a) $x^2 - 1 = 0$
(1b) $x^2 = 0$
(1c) $x^2 + 1 = 0$

For (1a), we can use the difference of squares formula, to factorize the left-hand side, and get $(x+1)(x-1) = 0$. Now, since we have two things which multiplied together give zero, we can conclude that at least one of them must be zero. This is because if you multiply two numbers which aren't zero the result won't be zero either. So we can say $x + 1 = 0$ or $x - 1 = 0$. This means that $x = -1$ or $x = 1$. Substituting back in, we find that these are both, in fact, solutions to the equation. We express our answer either by saying "$x = -1$ or $x = 1$" or using set notation: $x \in \{-1, 1\}$. Note: another way to solve this is to rewrite it as $x^2 = 1$ and take the square root of both sides. But you must remember to take both the positive and negative square roots, so you get $x = \pm 1$.

For (1b), we see that x multiplied by itself is zero. This is the case if and only if x is zero. So we write the answer as $x = 0$.

For (1c), what we have is a sum of squares, which we know can't be factorized. We can also rewrite it as $x^2 = -1$. Now, squaring any number gives us something greater than or equal to zero: the square of any positive number is positive; the square of zero is zero; and the square of any negative number is positive (e.g. $(-3)^2 = 9$). So there's no number which, when squared, would give us -1 (or any other negative number). So this equation has no solutions. We express this with "there are no solutions".

Every other quadratic equation will be similar to one of these cases. One way to see this is to complete the square and express the left hand side in what I call the canonical quadratic form, $l(x-m)^2 + n$. Note that when we're dealing with equations, we can divide both sides by a constant, so we can make the coefficient of x^2 be 1 if we want to. For example:
$2x^2 + 8x + 9 = 0$
$x^2 + 4x + \frac{9}{2} = 0$
$x^2 + 4x + 4 - 4 + \frac{9}{2} = 0$
$(x+2)^2 + \frac{1}{2} = 0$
This is the canonical form above, with $l = 1$, $m = 2$, and $n = \frac{1}{2}$. This resembles (1c), since the constant term is positive. If we rewrite it as $(x+2)^2 = -\frac{1}{2}$, we see that there are no solutions, because the left-hand side is a square, which must be zero or positive, but the right-hand side is negative.

Though completing the square will always work and make it easy to see whether there are any solutions, you don't always need to complete the square: sometimes you can factorize the quadratic expression without doing that. For example:
$(x + 7x + 12) = 0$
$(x+3)(x+4) = 0$
Therefore, either $x + 3 = 0$ or $x + 4 = 0$, so that $x = -3$ or $x = -4$.

Exercises

Solve each of the following quadratic equations.

#E1. $2x^2 + x - 10 = 0$

#E2. $x^2 + 5x = 0$

#E3. $3x^2 - 2 = 0$

#E4. $2x^2 + 5x + 8 = 0$

#E5. $9x^2 - 3x - 2 = 0$

#E6. $x^2 + 2x - 10 = 0$

#E7. $4x^2 - 12x + 9 = 0$

#E8. $4x^2 + 12x + 16 = 0$

#E9. $5 - 2x - 3x^2 = 0$

#E10. $\frac{1}{8}x^2 + x + 2 = 0$

Solutions

#E1. $2x^2 + x - 10 = 0$
$(2x + 5)(x - 2) = 0$
Either $2x + 5 = 0$ or $x - 2 = 0$.
Therefore, $x = -\frac{5}{2}$ or $x = 2$.

#E2. $x^2 + 5x = 0$
$x(x + 5) = 0$
Either $x = 0$ or $x + 5 = 0$.
Therefore, $x = 0$ or $x = -5$.

#E3. $3x^2 - 2 = 0$
$x^2 = \frac{2}{3}$
$x = \pm\sqrt{\frac{2}{3}}$
This is already fine as an answer. If you want you can then write $x = \sqrt{\frac{2}{3}}$ or $x = -\sqrt{\frac{2}{3}}$, but it's not necessary since it's already expressed in the line with the $\pm$.

#E4. $2x^2 + 5x + 8 = 0$
$x^2 + \frac{5}{2}x + 4 = 0$
$x^2 + \frac{5}{2}x + \frac{25}{16} - \frac{25}{16} + 4 = 0$
$(x + \frac{5}{4})^2 + \frac{39}{4} = 0$
So this has no solutions (we can rewrite it as $(x + \frac{5}{4})^2 = -\frac{39}{4}$ to see that).

#E5. $9x^2 - 3x - 2 = 0$
$(3x - 2)(3x + 1) = 0$
Either $3x - 2 = 0$ or $3x + 1 = 0$.
Therefore, $x = \frac{2}{3}$ or $x = -\frac{1}{3}$.

#E6. $x^2 + 2x - 10 = 0$
$x^2 + 2x + 1 - 1 - 10 = 0$
$(x + 1)^2 - 11 = 0$
$(x + 1 + \sqrt{11})(x + 1 - \sqrt{11}) = 0$
Either $x + 1 + \sqrt{11} = 0$ or $x + 1 - \sqrt{11} = 0$.
Therefore $x = -1 - \sqrt{11}$ or $x = -1 + \sqrt{11}$.

#E7. $4x^2 - 12x + 9 = 0$
$(2x - 3)^2 = 0$
$2x - 3 = 0$
Therefore, $x = \frac{3}{2}$.

#E8. $4x^2 + 12x + 16 = 0$
$x^2 + 3x + 4 = 0$
$x^2 + 3x + \frac{9}{4} - \frac{9}{4} + 4 = 0$
$(x + \frac{3}{2})^2 + \frac{7}{4} = 0$
There are no solutions (since the left hand side is a square plus a positive number, it's always a positive number).

#E9. $5 - 2x - 3x^2 = 0$
$3x^2 + 2x - 5 = 0$
$(3x + 5)(x - 1) = 0$
Either $3x + 5 = 0$ or $x - 1 = 0$.
Therefore, $x = -\frac{5}{3}$ or $x = 1$.

#E10. $\frac{1}{8}x^2 + x + 2 = 0$
$x^2 + 8x + 16 = 0$
$(x + 4)^2 = 0$
Therefore $x + 4 = 0$, so that $x = -4$.