# Polynomial equations

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Here are a couple of worked examples of polynomial equation:
A. $2x^3 - 4x^2 + 10x - 20 = 0$
We've only done equations where the leading coefficient (the coefficient of the highest order term) was 1. For this equation it's 2. However, since all coefficients are divisible by 2, we can start by dividing by 2 on both sides of the equation:
$x^3 - 2x^2 + 5x - 10 = 0$
Now, we can use the rational roots theorem, to say that the possible rational roots are $1, -1, 2, -2, 5, -5, 10, -10$. We proceed to test them until we find one which is a root. We can use either substitution or synthetic (or polynomial division). Here I'll do substitution.
Testing $x = 1$: substitute $x = 1$, and we get $1 - 2 + 5 - 10 = -6$, not zero, so $x = 1$ isn't a root
Testing $x = 11$: substitute $x = 11$, and we get $-1 - 2 - 5 - 10 = -18$, not zero, so $x = 1$ isn't a root
Testing $x = 2$: substitute $x = 2$, and we get $8 - 8 + 10 - 10 = 0$. So $x = 2$ is a root.
Now, we could either try the other ones, or divide our polynomial by $x - 2$ to see what's left. The safest way is doing the division, which gives $\frac{x^3-2x^2+5x-10}{x-2} = x^2 + 1$. So that $x^3 - 2x^2 + 5x - 10 = (x - 2)(x^2 + 1)$.
So, since $x^-3 - 2x^2 + 5x - 10 = 0$, $(x - 2)(x^2 + 1) = 0$, so that either $x - 2 = 0$ or $x^2 + 1 = 0$. But we know $x^2 + 1 = 0$ has no solutions. So that means the only solution is $x = 2$.

B. $x^4 - 2x^3 + x - 2 = 0$
There are several ways to proceed. One is to factorize this by grouping: $x^4 - 2x^3 + x - 2 = (x^4 + x) - (2x^3 + 2) = x(x^3 + 1) - 2(x^3 + 1) = (x - 2)(x^3 + 1)$. And we can then continue from there. You can factorize $x^3+1$ using the sum of cubes formula to get $x^3+1 = (x-1)(x^2-x+1)$. This means $x = 1$ is a root, and if you then work with $x^2 - x + 1$ (either completing the square or by fiding the discriminant) you will find it has no solutions. So the solutions to the original equation are $x = 2$ and $x = 1$.
However, if you don't see that it's possible to use grouping there, you can use the rational roots theorem and proceed to test the divisors of -2: $1, -1, 2, -2$.
Testing $x = 1$, we substitute and get $1 - 2 + 1 - 2 = 0$, so $x = 1$ is a solution. We can then do the division of the polynomial by $x - 1$ and we get $x^4 - 2x^3 + x - 2 = (x - 1)(x^3 - 3x^2 + 3x - 2$. So either $x = 1$ or $x^3 - 3x^2 + 3x - 2 = 0$. So let's try to find the solutions for this cubic equation. Again, we can test the same roots.
Note: we have to test $x = 1$ again, because it's possible that the polynomial could be divisible by $(x - 1)$.
Testing $x = 1$, we get by substituting $1 - 3 + 3 - 2 = -1$, not zero, so $x = 1$ is not a solution of this cubic equation (though it is a solution of our original equation).
Testing $x = -1$, we get $-1 - 3 + 3 - 2 = -3$, not zero, so $x = -1$ is not a solution.
Testing $x = 2$, we get $8 - 12 + 6 - 2 = 0$, so $x = 2$ is a solution.
We then divide $x^3 - 3x^2 + 3x - 2$ by $x - 2$, which gives us $x^2 - x + 1$, which (as mentioned above) is a quadratic without any solutions.
So the solutions to our original equation are $x = 1$ and $x = 2$.